

# **COMP I 22/L Lecture I 9**

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# Overview

- Circuit minimization
  - Boolean algebra
  - Karnaugh maps

# Circuit Minimization

# Motivation

- Unnecessarily large programs: bad
- Unnecessarily large circuits: Very Bad™
  - Why?

# Motivation

- Unnecessarily large programs: bad
- Unnecessarily large circuits: Very Bad™
  - Why?
    - Bigger circuits = bigger chips = higher cost (non-linear too!)
    - Longer circuits = more time needed to move electrons through = slower

# Simplification

- Real-world formulas can often be simplified, according to algebraic rules
  - How might we simplify the following?

$$R = A * B + !A * B$$

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- Real-world formulas can often be simplified, according to algebraic rules
  - How might we simplify the following?

$$R = A * B + !A * B$$

$$R = B(A + !A)$$

$$R = B(\text{true})$$

$$R = B$$

# Simplification Trick

- Look for products that differ only in one variable
  - One product has the original variable ( $A$ )
  - The other product has the other variable ( $\neg A$ )

$$R = \textcolor{red}{A * B} + \textcolor{red}{\neg A * B}$$

# Additional Example I

$!ABCD + ABCD + !AB!CD + AB!CD$

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$$\begin{aligned} & !ABCD + ABCD + !AB!CD + AB!CD \\ & BCD(A + !A) + !AB!CD + AB!CD \end{aligned}$$

# Additional Example I

$$!ABCD + ABCD + !AB!CD + AB!CD$$
$$BCD(A + !A) + !AB!CD + AB!CD$$
$$\textcolor{red}{BCD} + !AB!CD + AB!CD$$

# Additional Example I

$$!ABCD + ABCD + !AB!CD + AB!CD$$

$$BCD(A + !A) + !AB!CD + AB!CD$$

$$BCD + !AB!CD + AB!CD$$

$$BCD + B!CD(!A + A)$$

# Additional Example I

$!ABCD + ABCD + !AB!CD + AB!CD$

$BCD(A + !A) + !AB!CD + AB!CD$

$BCD + !AB!CD + AB!CD$

$BCD + B!CD(!A + A)$

$BCD + B!CD$

# Additional Example I

$!ABCD + ABCD + !AB!CD + AB!CD$

$BCD(A + !A) + !AB!CD + AB!CD$

$BCD + !AB!CD + AB!CD$

$BCD + B!CD(!A + A)$

$BCD + B!CD$

$BD(C + !C)$

# Additional Example I

$!ABCD + ABCD + !AB!CD + AB!CD$

$BCD(A + !A) + !AB!CD + AB!CD$

$BCD + !AB!CD + AB!CD$

$BCD + B!CD(!A + A)$

$BCD + B!CD$

$BD(C + !C)$

$BD$

# Additional Example 2

$!A!BC + A!B!C + !ABC + !AB!C + A!BC$

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$$!A!BC + A!B!C + !ABC + !AB!C + A!BC$$
$$!A!BC + A!BC + A!B!C + !ABC + !AB!C$$

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$$!A!BC + A!B!C + !ABC + !AB!C + A!BC$$

$$!A!BC + A!BC + A!B!C + !ABC + !AB!C$$

$$!BC(A + !A) + A!B!C + !ABC + !AB!C$$

# Additional Example 2

$!A!BC + A!B!C + !ABC + !AB!C + A!BC$

$!A!BC + A!BC + A!B!C + !ABC + !AB!C$

$!BC(A + !A) + A!B!C + !ABC + !AB!C$

$\textcolor{red}{!BC} + A!B!C + !ABC + !AB!C$

# Additional Example 2

$$!A!BC + A!B!C + !ABC + !AB!C + A!BC$$

$$!A!BC + A!BC + A!B!C + !ABC + !AB!C$$

$$!BC(A + !A) + A!B!C + !ABC + !AB!C$$

$$!BC + A!B!C + !ABC + !AB!C$$

$$!BC + A!B!C + !AB(C + !C)$$

# Additional Example 2

$$!A!BC + A!B!C + !ABC + !AB!C + A!BC$$

$$!A!BC + A!BC + A!B!C + !ABC + !AB!C$$

$$!BC(A + !A) + A!B!C + !ABC + !AB!C$$

$$!BC + A!B!C + !ABC + !AB!C$$

$$!BC + A!B!C + !AB(C + !C)$$

$$!BC + A!B!C + \textcolor{red}{!AB}$$

# De Morgan's Laws

Also potentially useful for simplification

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$$! (A + B) = !A !B$$

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$$! (A + B) = !A !B$$

$$! (AB) = !A + !B$$

# De Morgan Example

$$! (X + Y) ! (!X + Z)$$

# De Morgan Example

$$! (X + Y) ! (!X + Z)$$

!A

!B

# De Morgan Example

$$! (X + Y) ! (!X + Z)$$

!A

!B

# De Morgan Example

$$! (X + Y) ! (!X + Z)$$

$!A$

$!B$

From De Morgan's Law:

$$! (A + B) = !A !B$$

# De Morgan Example

$$! (X + Y) ! (!X + Z)$$

$!A$

$!B$

From De Morgan's Law:

$$! (A + B) = !A !B$$

$$! (X + Y + !X + Z)$$

# De Morgan Example

$$! (X + Y) ! (!X + Z)$$

!A

!B

From De Morgan's Law:

$$! (A + B) = !A !B$$

$$! (X + Y + !X + Z)$$

$$! (X + !X + Y + Z)$$

# De Morgan Example

$$! (X + Y) ! (!X + Z)$$

$!A$

$!B$

From De Morgan's Law:

$$! (A + B) = !A !B$$

$$! (X + Y + !X + Z)$$

$$! (X + !X + Y + Z)$$

$$! (\text{true} + Y + Z)$$

# De Morgan Example

$$! (X + Y) ! (!X + Z)$$

$!A$

$!B$

From De Morgan's Law:

$$! (A + B) = !A !B$$

$$! (X + Y + !X + Z)$$

$$! (X + !X + Y + Z)$$

$$! (\text{true} + Y + Z)$$

$$! (\text{true})$$

# De Morgan Example

$$! (X + Y) ! (!X + Z)$$

!A

!B

From De Morgan's Law:

$$! (A + B) = !A !B$$

$$! (X + Y + !X + Z)$$

$$! (X + !X + Y + Z)$$

$$! (\text{true} + Y + Z)$$

$$! (\text{true})$$

false

# Scaling Up

- Performing this sort of algebraic manipulation by hand can be tricky
- We can use *Karnaugh maps* to make it immediately apparent as to what can be simplified

# Example

$$R = A^*B + !A^*B$$

# Example

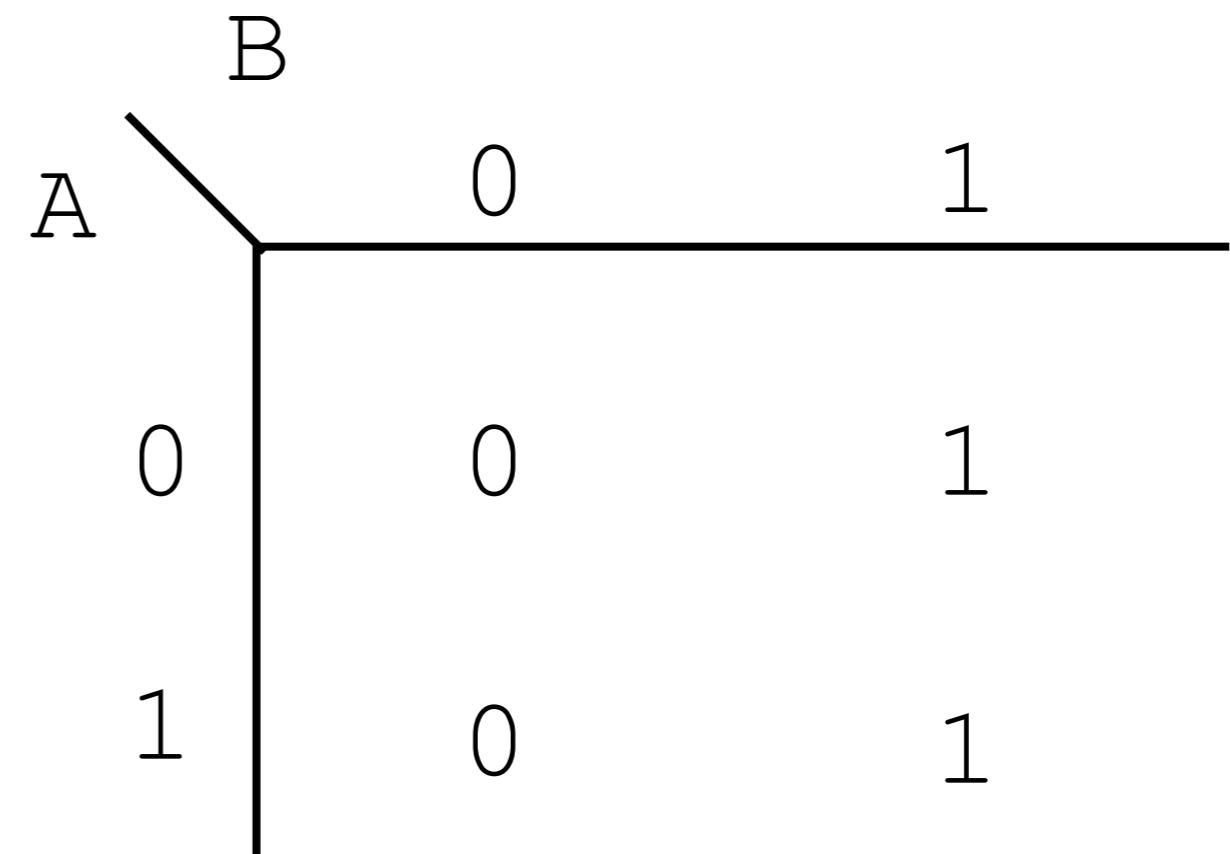
$$R = A^*B + !A^*B$$

A	B	O
0	0	0
0	1	1
1	0	0
1	1	1

# Example

$$R = A^*B + !A^*B$$

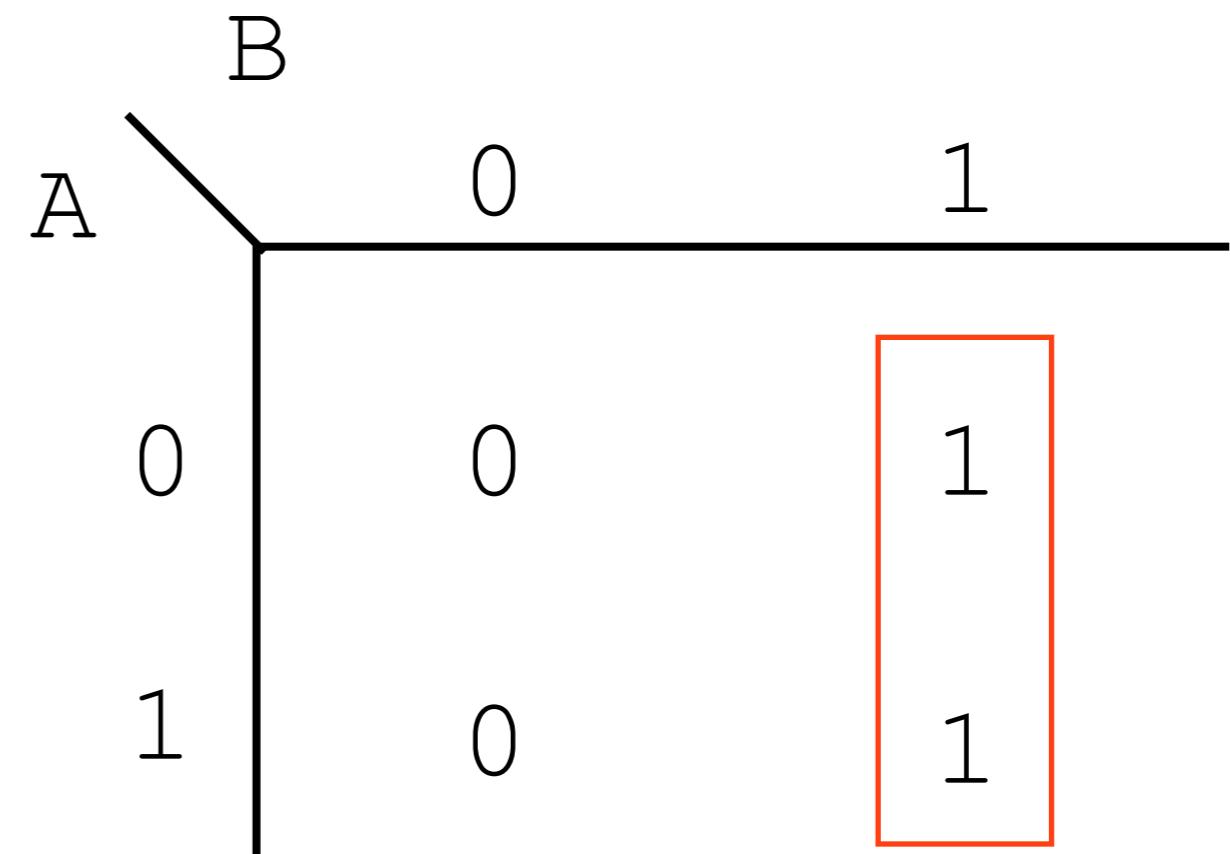
A	B	O
0	0	0
0	1	1
1	0	0
1	1	1



# Example

$$R = A^*B + !A^*B$$

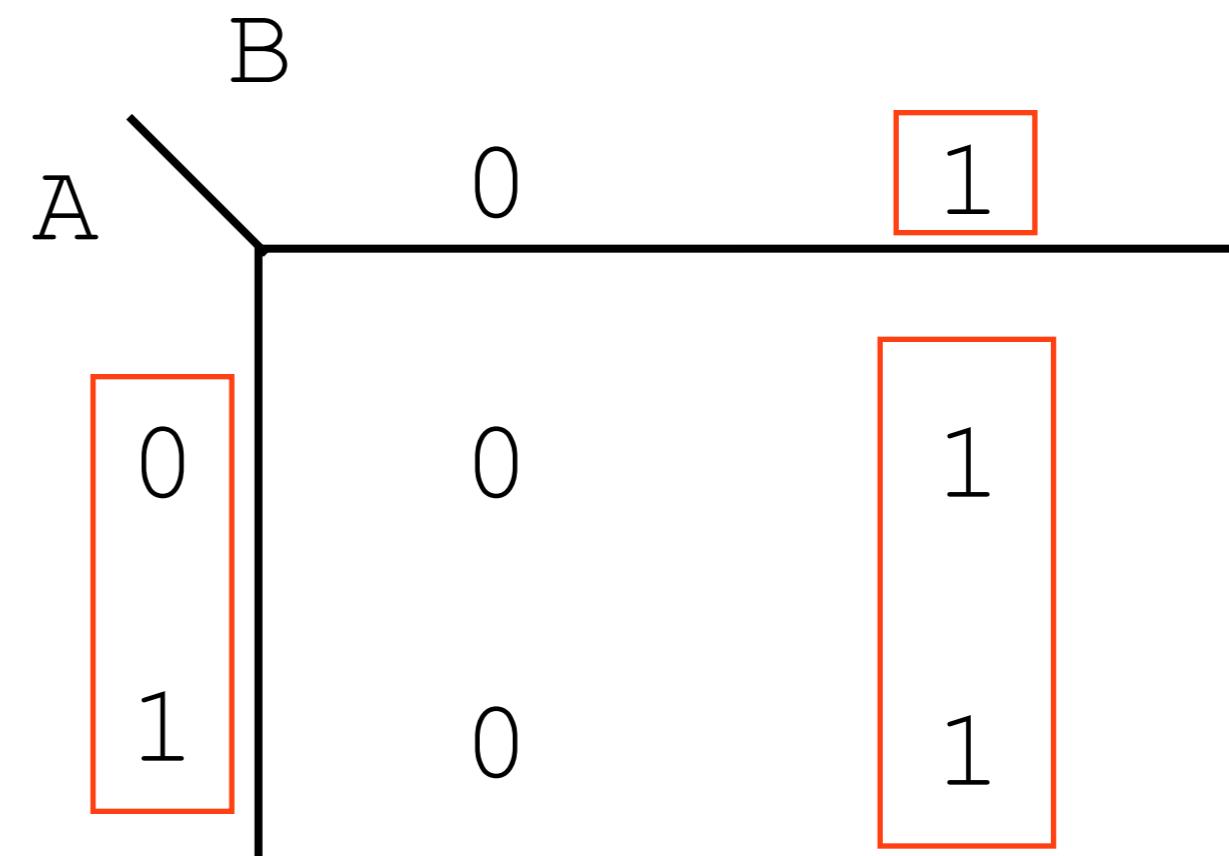
A	B	O
0	0	0
0	1	1
1	0	0
1	1	1



# Example

$$R = A^*B + !A^*B$$

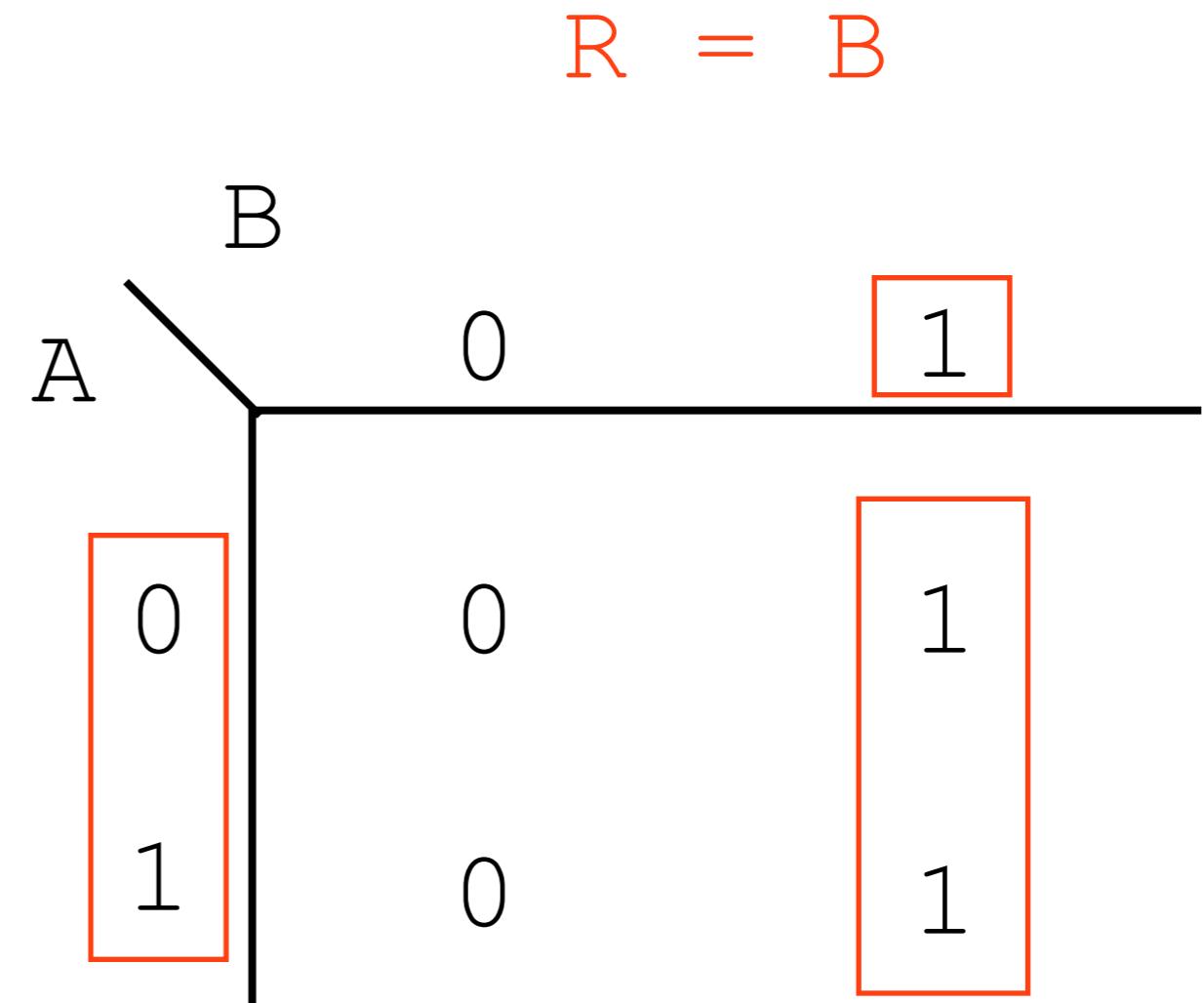
A	B	O
0	0	0
0	1	1
1	0	0
1	1	1



# Example

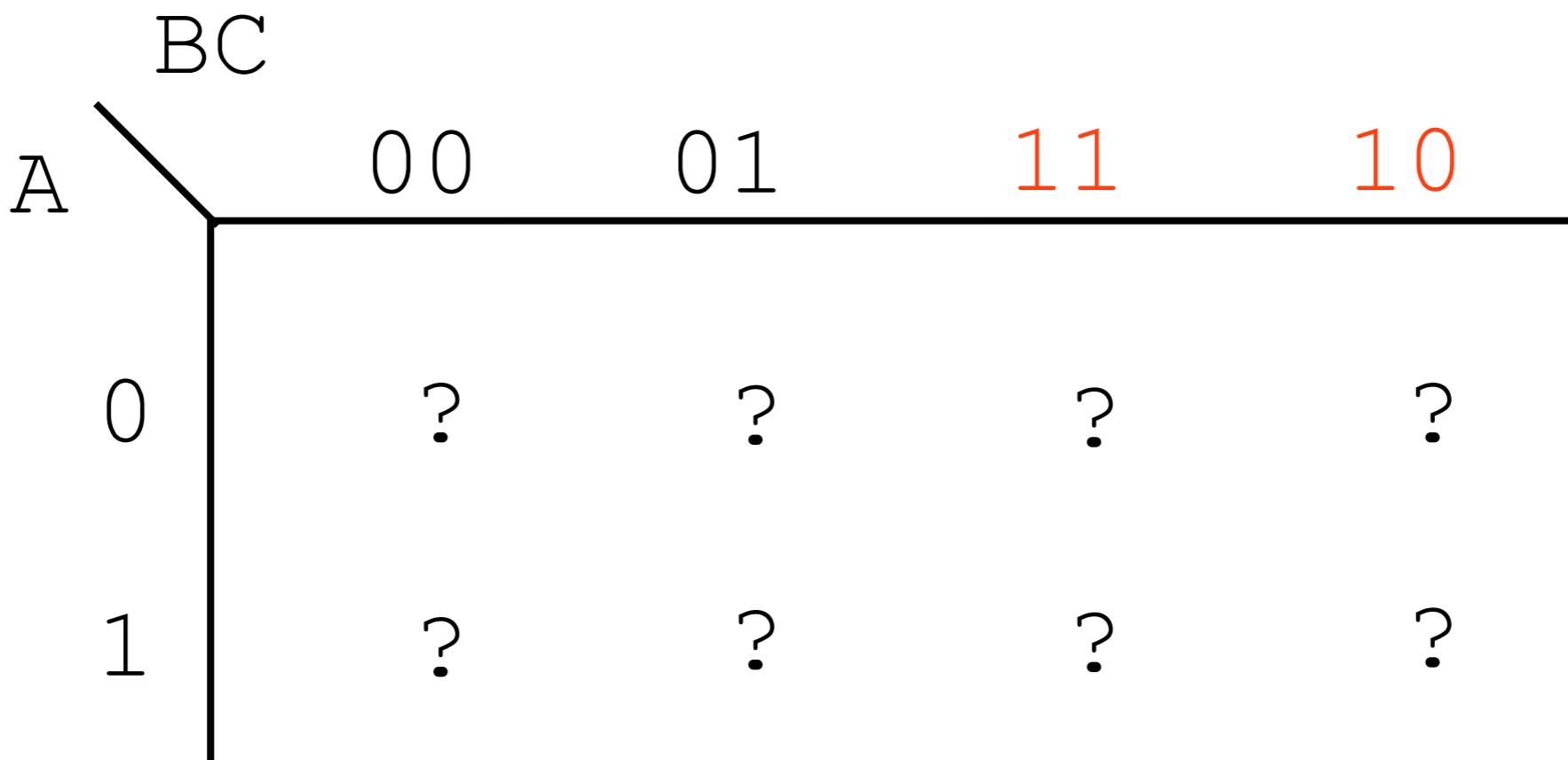
$$R = A^*B + !A^*B$$

A	B	O
0	0	0
0	1	1
1	0	0
1	1	1



# Three Variables

- We can scale this up to three variables, by combining two variables on one axis
- The combined axis must be arranged such that only one bit changes per position



# Three Variable Example

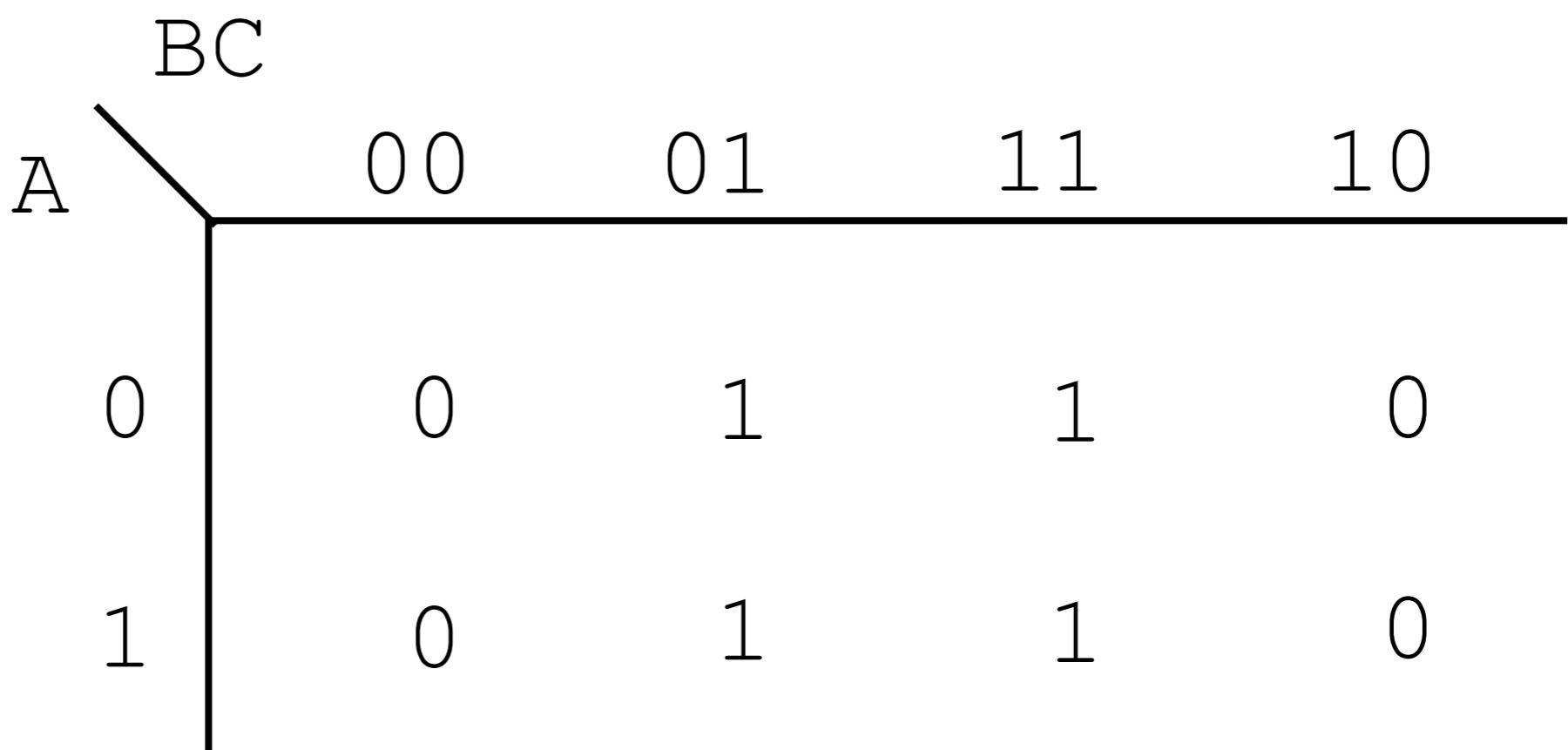
$$R = !A!BC + !ABC + A!BC + ABC$$

$$R = !A!BC + !ABC + A!BC + ABC$$

A	B	C	R
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

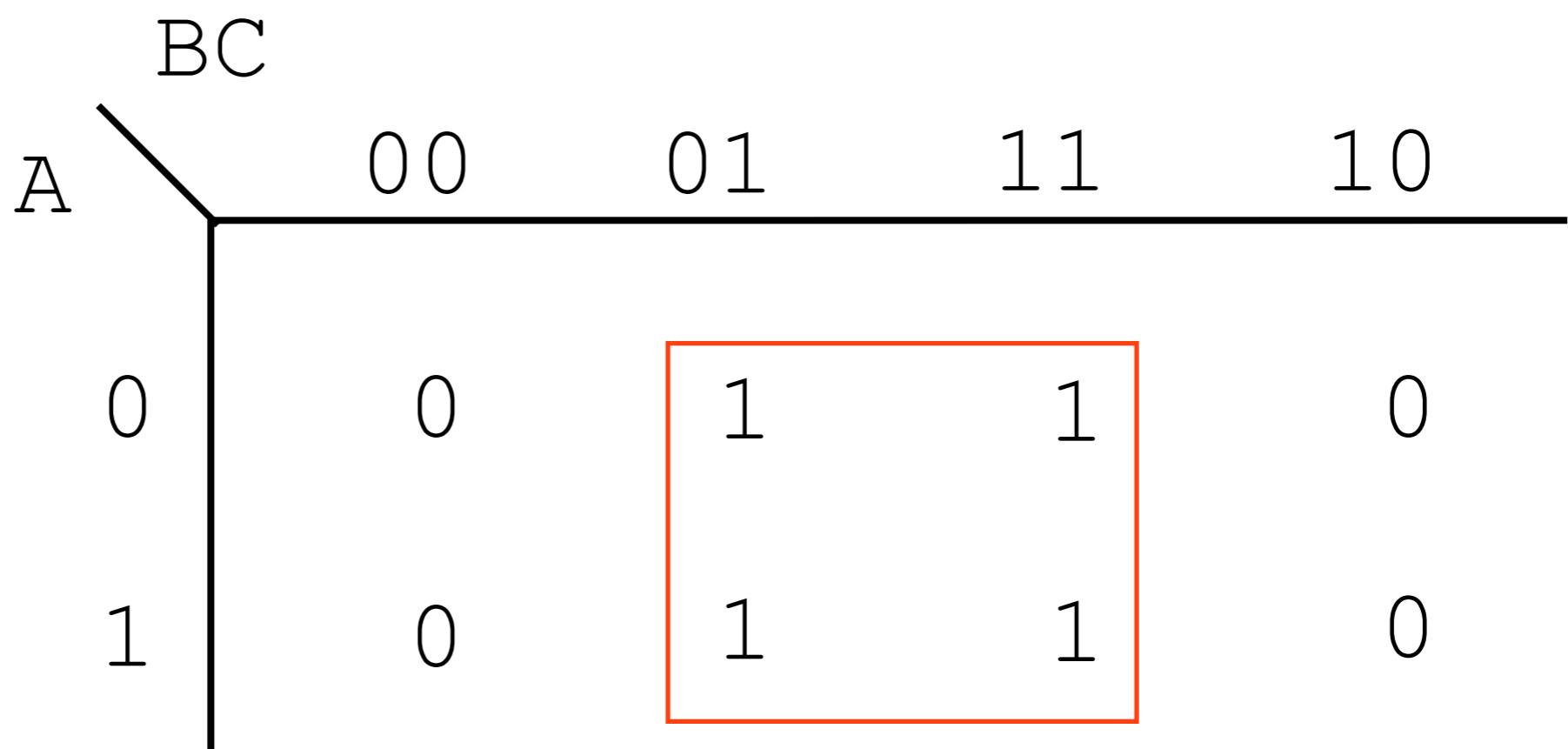
$$R = !A!BC + !ABC + A!BC + ABC$$

A	B	C	R
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1



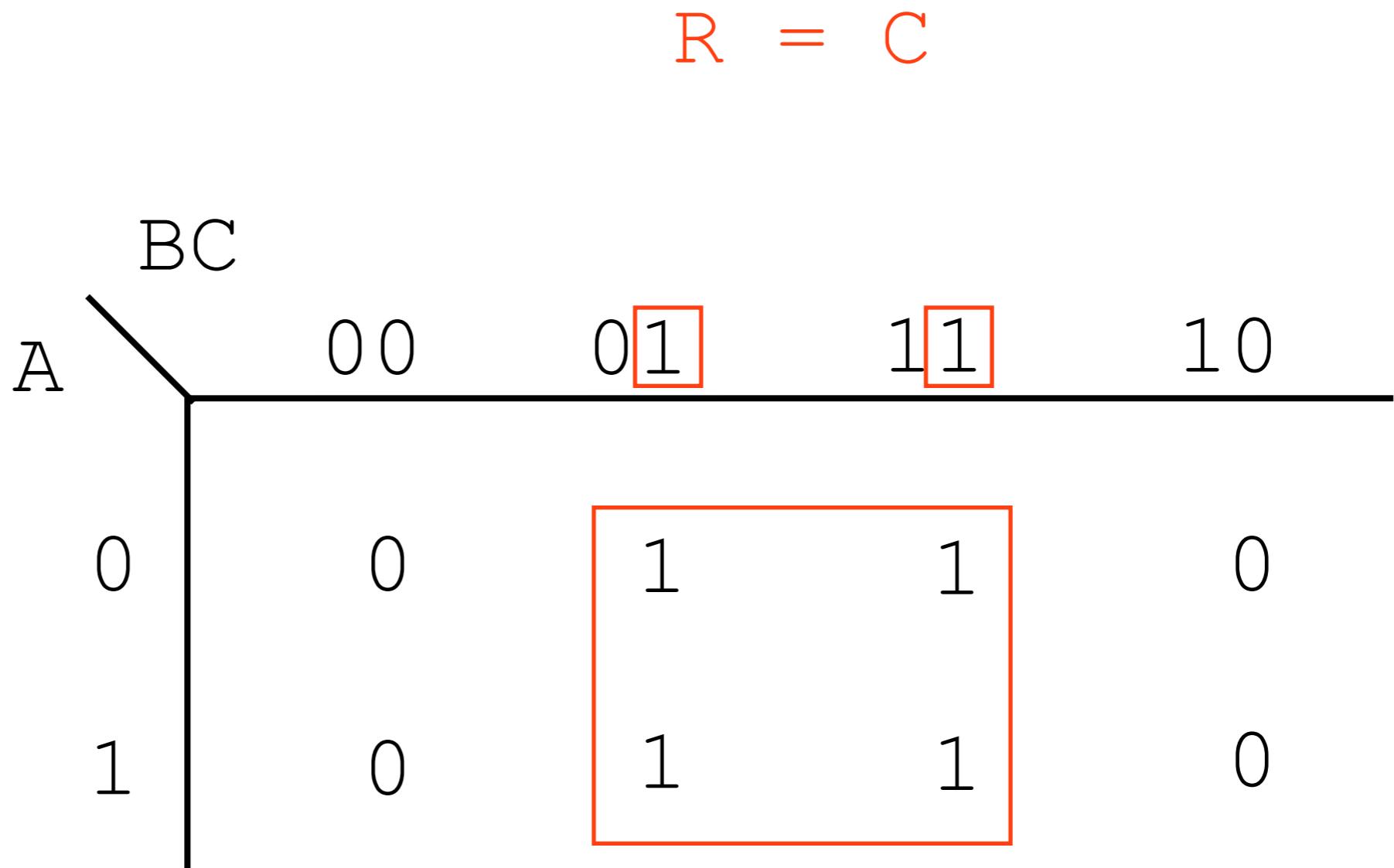
$$R = !A!BC + !ABC + A!BC + ABC$$

A	B	C	R
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1



$$R = !A!BC + !ABC + A!BC + ABC$$

A	B	C	R
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1



# **Another Three Variable Example**

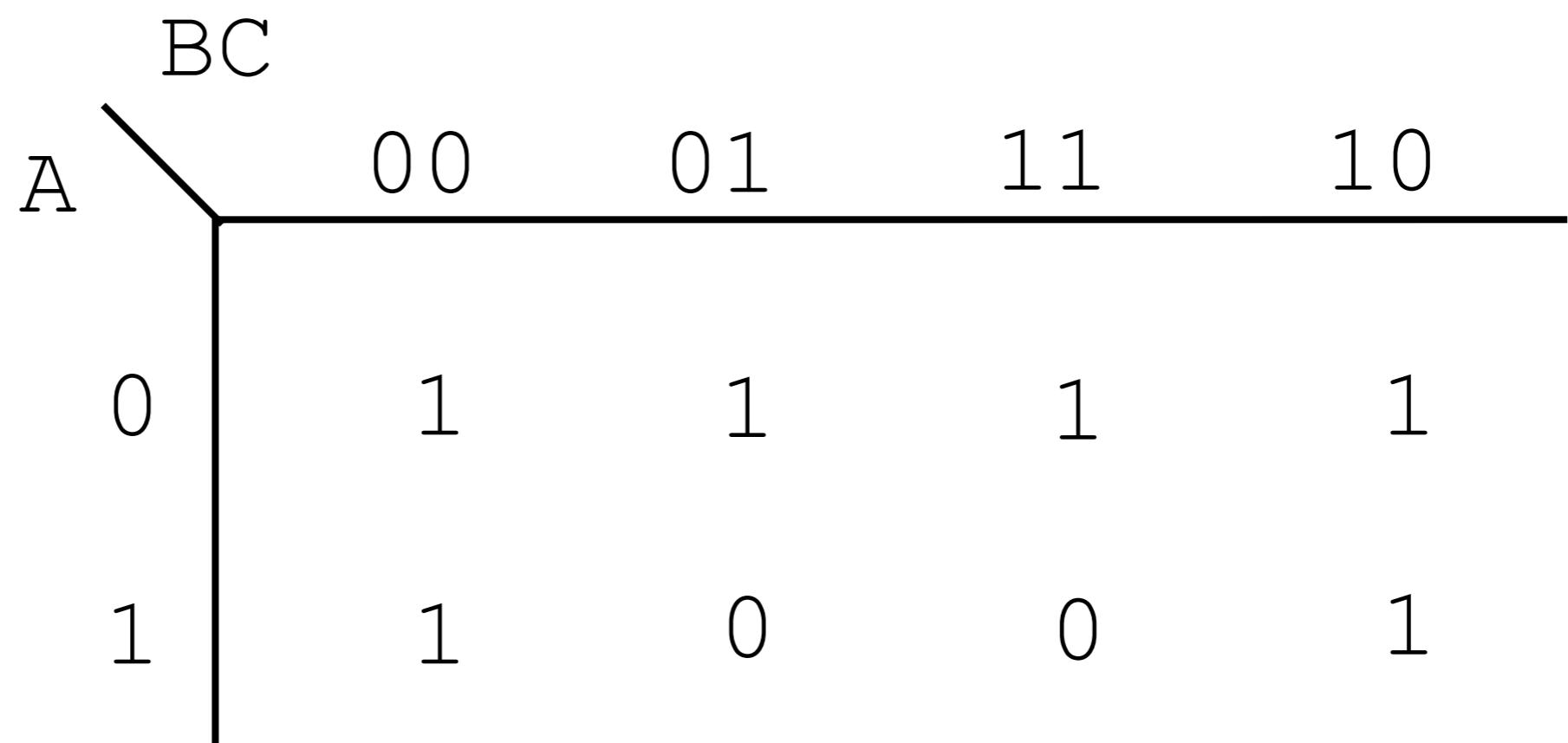
$$\begin{aligned} R = & \quad !A!B!C + !A!BC + !ABC + \\ & !AB!C + A!B!C + AB!C \end{aligned}$$

$$\begin{aligned} R = & \quad !A!B!C + !A!BC + !ABC + \\ & !AB!C + A!B!C + AB!C \end{aligned}$$

A	B	C	R
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

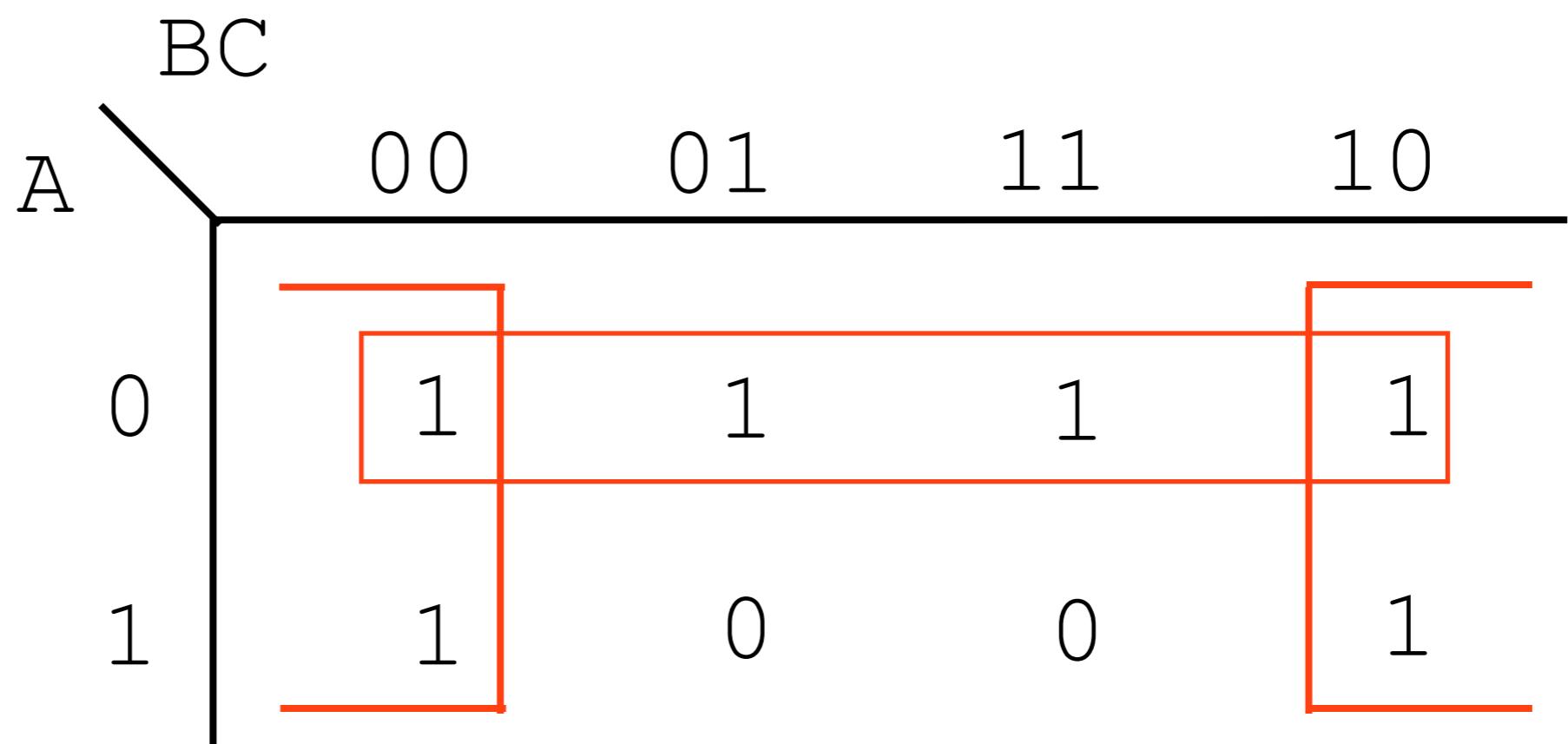
$$R = !A!B!C + !A!BC + !ABC + \\ !AB!C + A!B!C + AB!C$$

A	B	C	R
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0



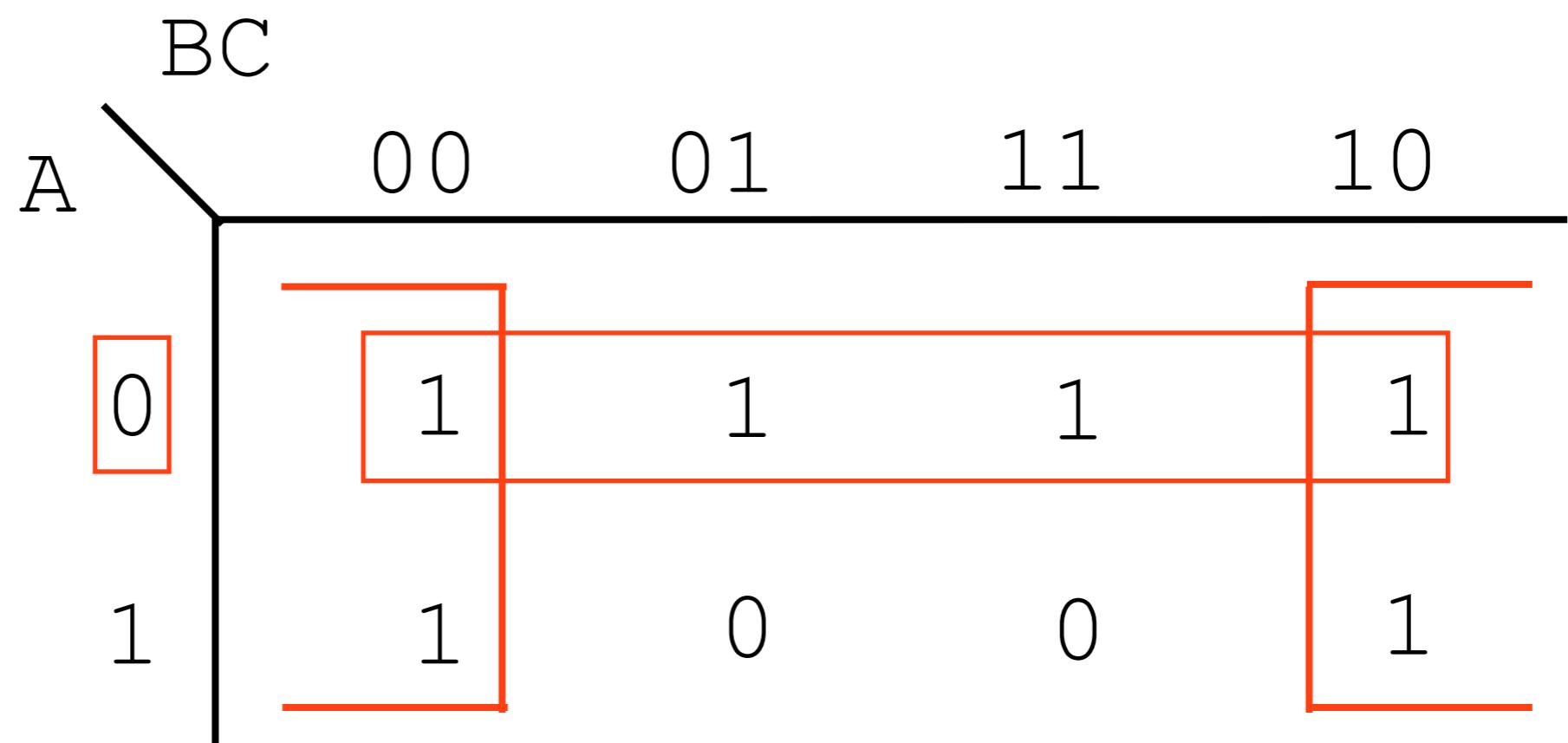
$$R = !A!B!C + !A!BC + !ABC + \\ !AB!C + A!B!C + AB!C$$

A	B	C	R
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0



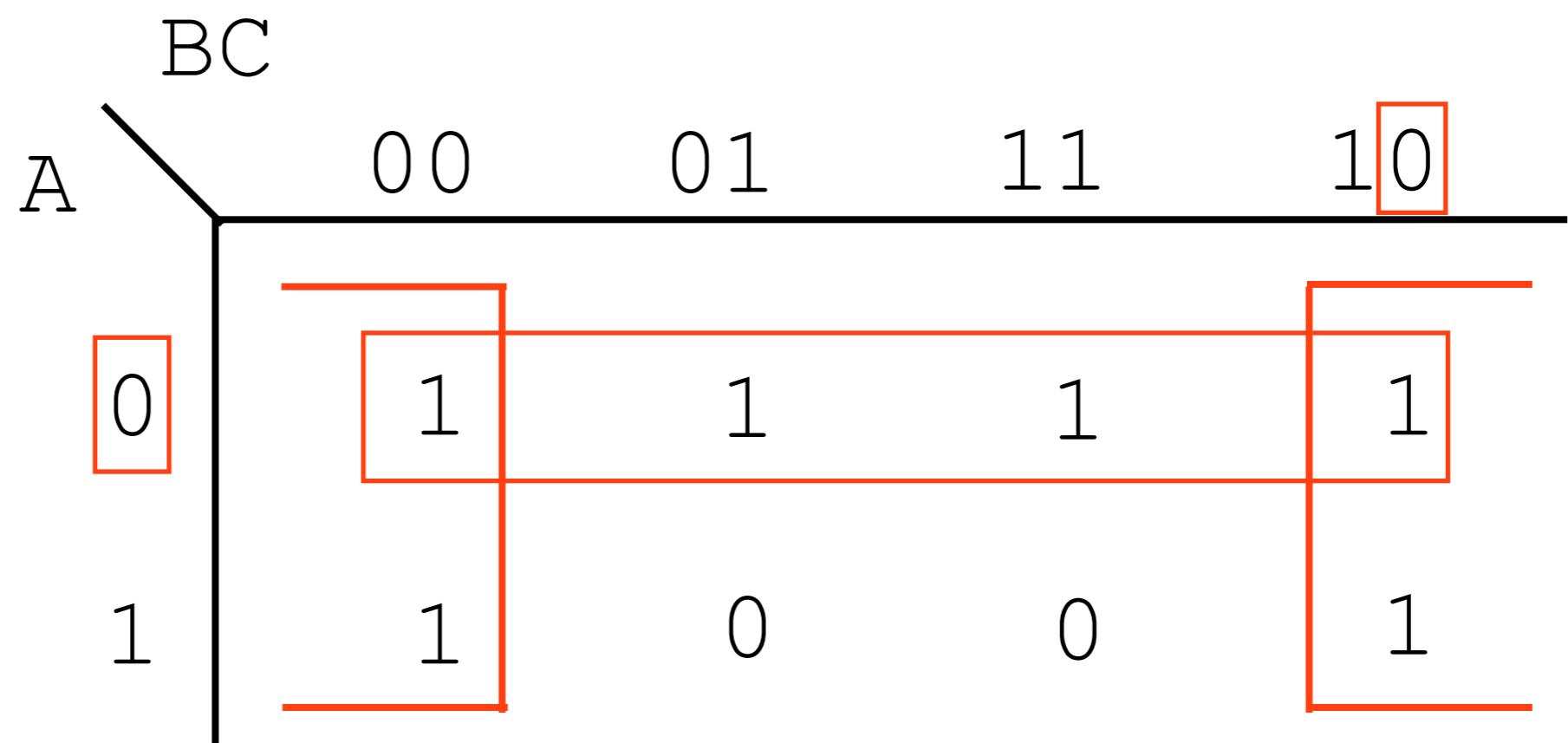
$$R = !A!B!C + !A!BC + !ABC + \\ !AB!C + A!B!C + AB!C$$

A	B	C	R
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0



$$R = !A!B!C + !A!BC + !ABC + \\ !AB!C + A!B!C + AB!C$$

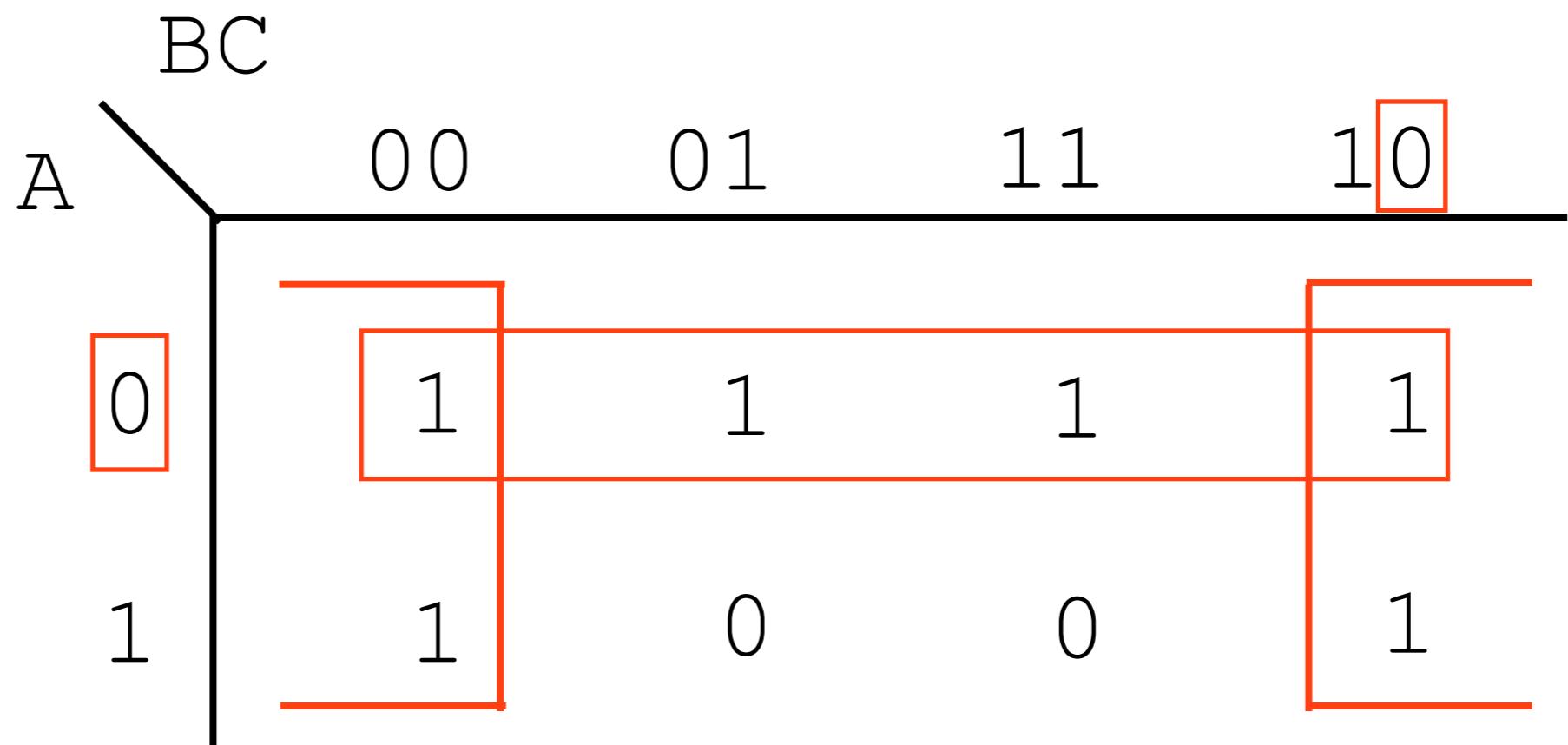
A	B	C	R
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0



$$R = !A!B!C + !A!BC + !ABC + \\ !AB!C + A!B!C + AB!C$$

A	B	C	R
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

$$R = !A + !C$$



# Four Variable Example

$$R = !A!B!C!D + !A!B!CD + !A!BC!D + \\ !ABC!D + A!B!C!D + A!B!CD + A!BC!D$$

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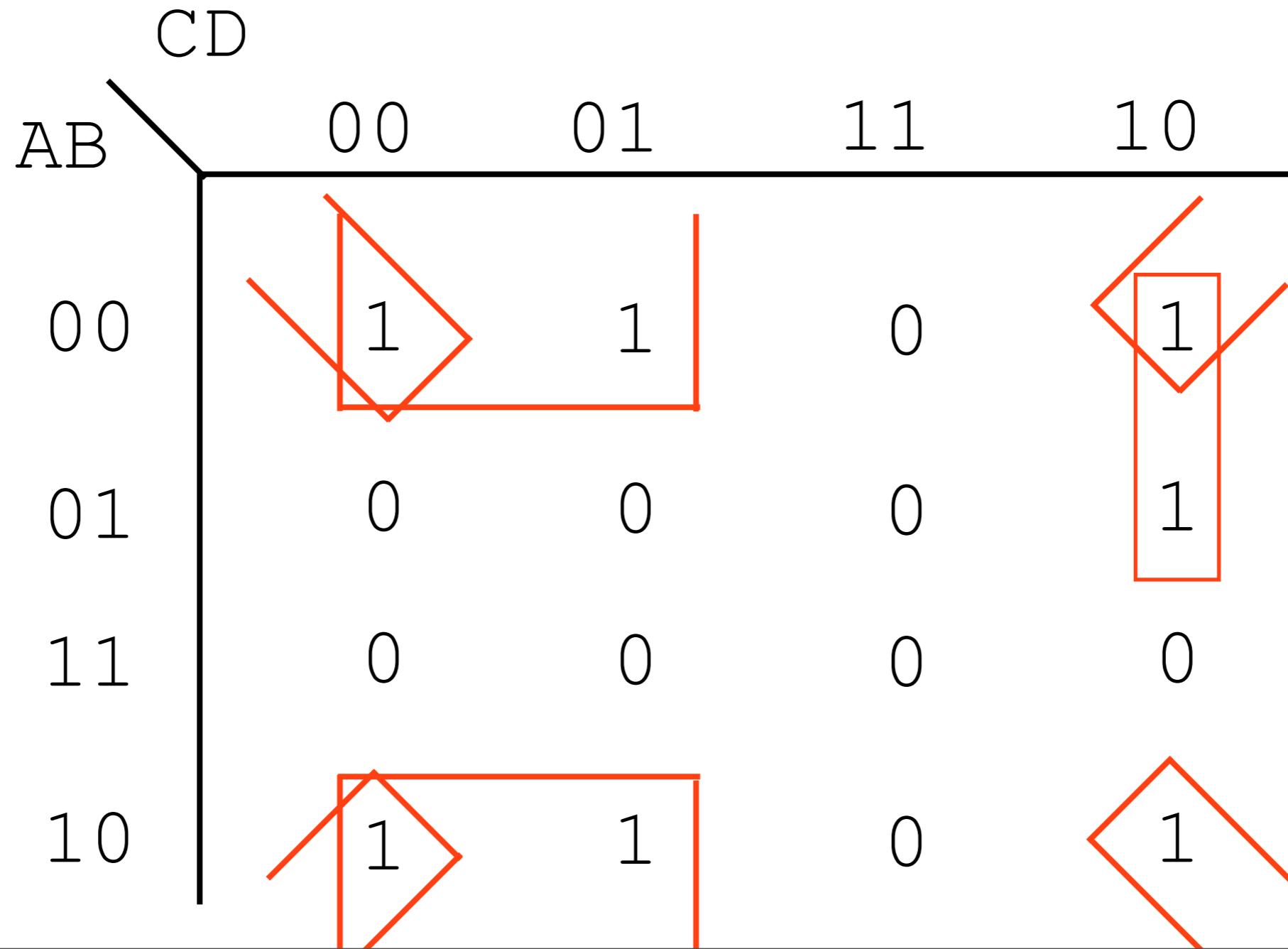
$$R = !A!B!C!D + !A!B!CD + !A!BC!D + \\ !ABC!D + A!B!C!D + A!B!CD + A!BC!D$$

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		CD	00	01	11	10
		AB	00	01	11	10
AB	00	1	1	0	1	
	01	0	0	0	1	
	11	0	0	0	0	
	10	1	1	0	1	

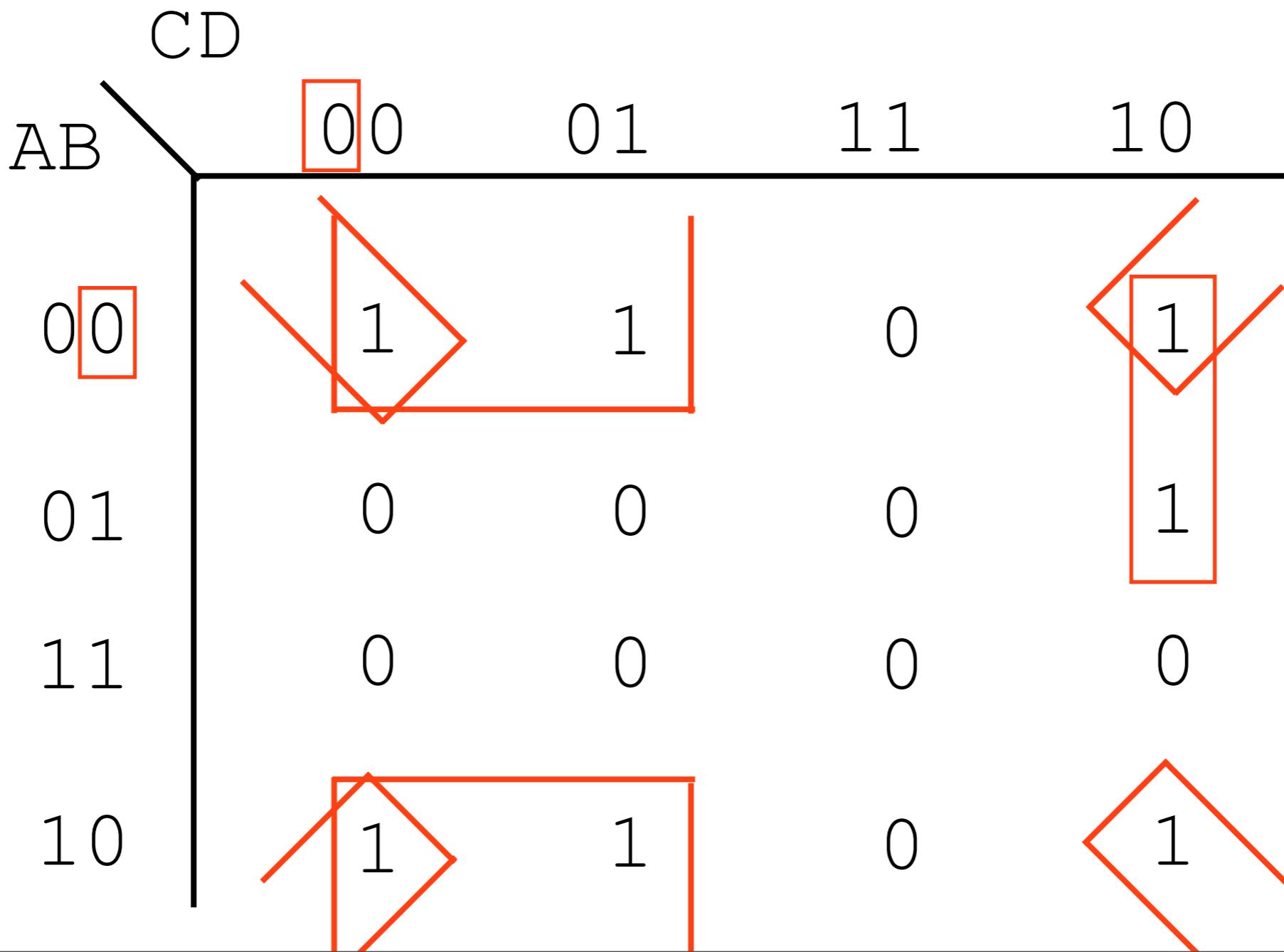
$$R = !A!B!C!D + !A!B!CD + !A!BC!D + \\ !ABC!D + A!B!C!D + A!B!CD + A!BC!D$$

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$$R = !A!B!C!D + !A!B!CD + !A!BC!D + \\ !ABC!D + A!B!C!D + A!B!CD + A!BC!D$$

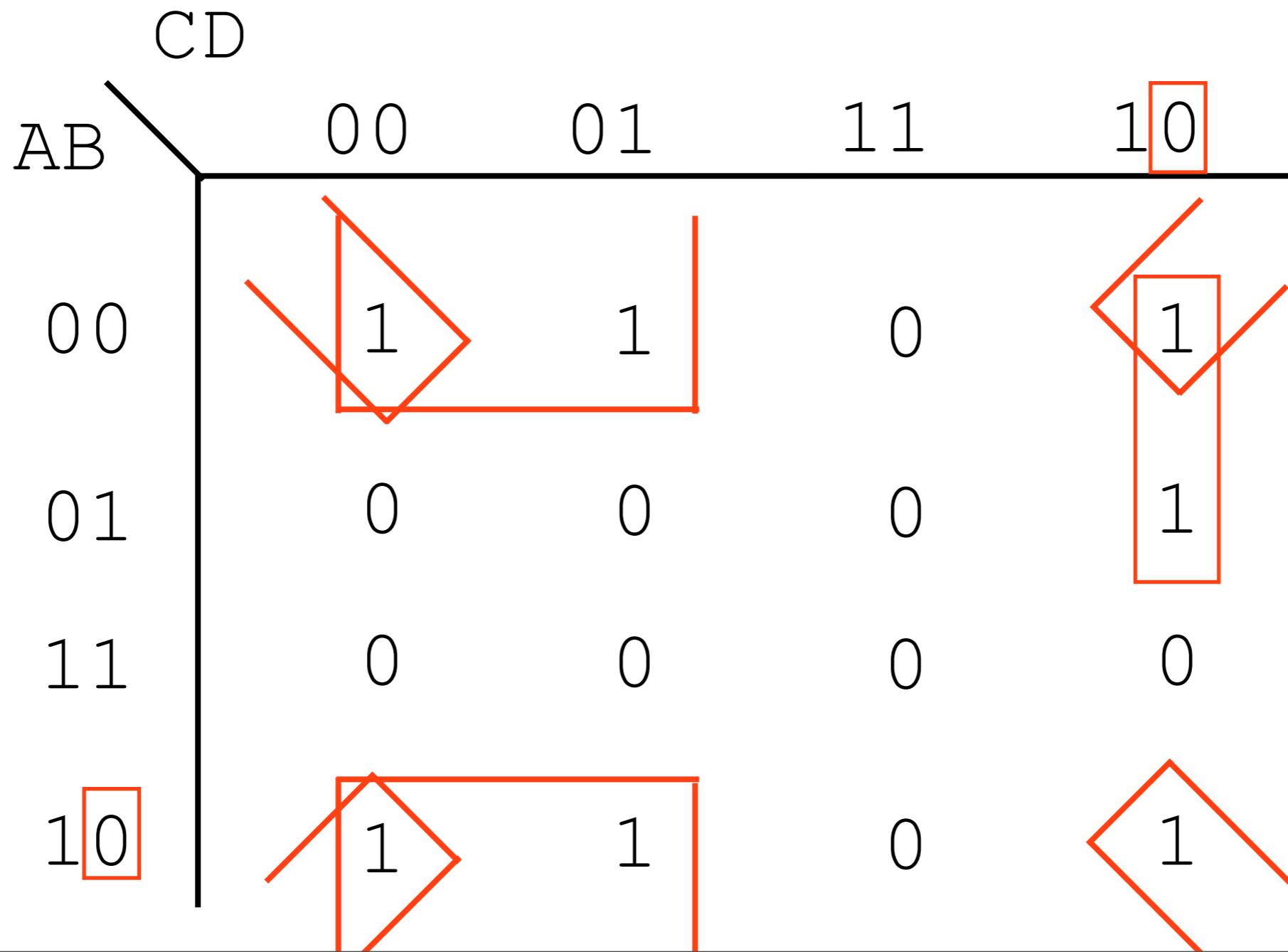
$$R = !B!C$$



$$R = !A!B!C!D + !A!B!CD + !A!BC!D + \\ !ABC!D + A!B!C!D + A!B!CD + A!BC!D$$

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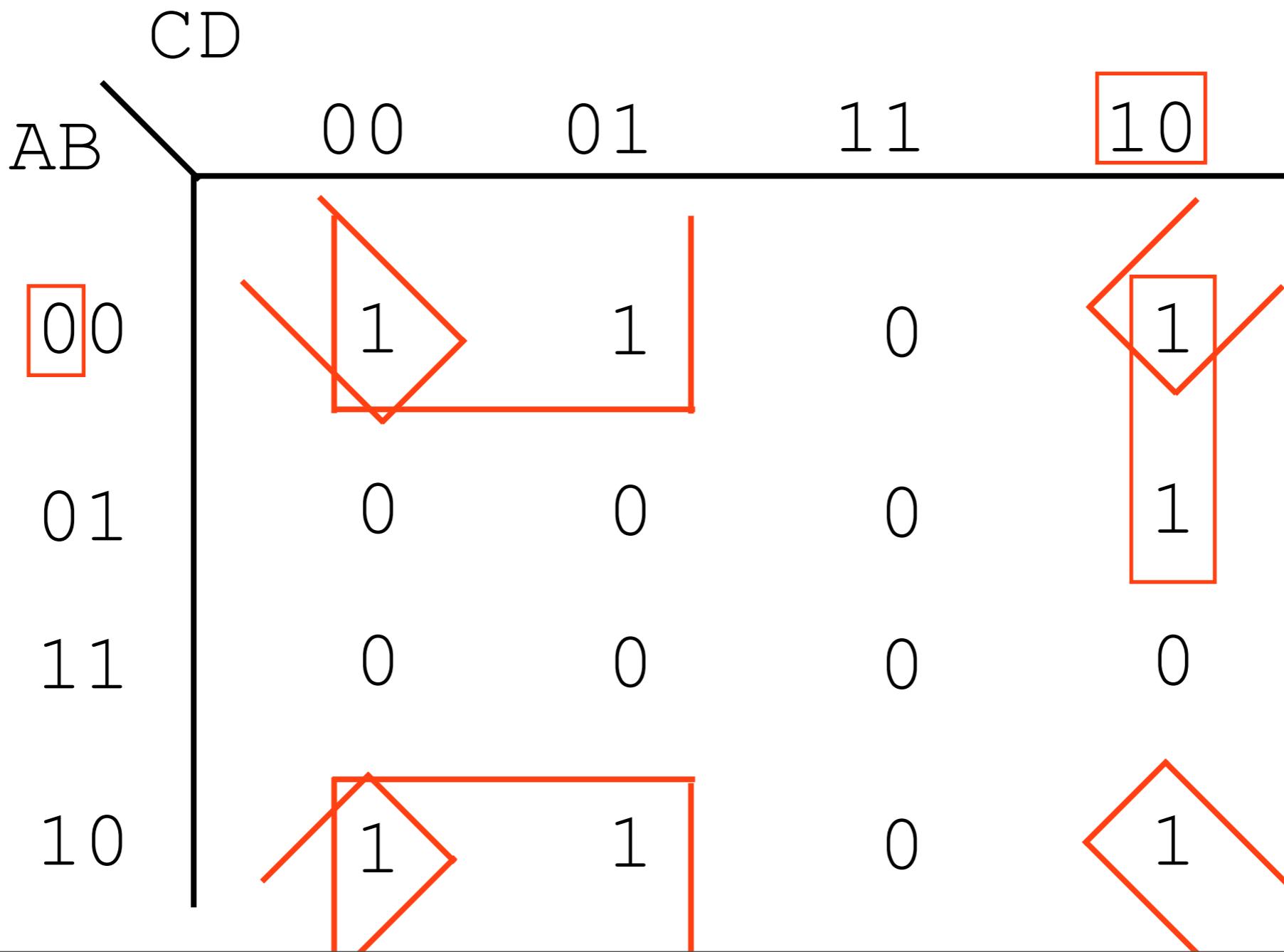
$$R = !B!C + !B!D$$



$$R = !A!B!C!D + !A!B!CD + !A!BC!D + \\ !ABC!D + A!B!C!D + A!B!CD + A!BC!D$$

---

$$R = !B!C + !B!D + !AC!D$$



# K-Map Rules in Summary (I)

- Groups can contain only 1s
- Only 1s in adjacent groups are allowed (no diagonals)
- The number of 1s in a group must be a power of two (1, 2, 4, 8...)
- The groups must be as large as legally possible

# K-Map Rules in Summary (2)

- All 1s must belong to a group, even if it's a group of one element
- Overlapping groups are permitted
- Wrapping around the map is permitted
- Use the fewest number of groups possible

# Revisiting Problem

$!A!BC + A!B!C + !ABC + !AB!C + A!BC$

# Revisiting Problem

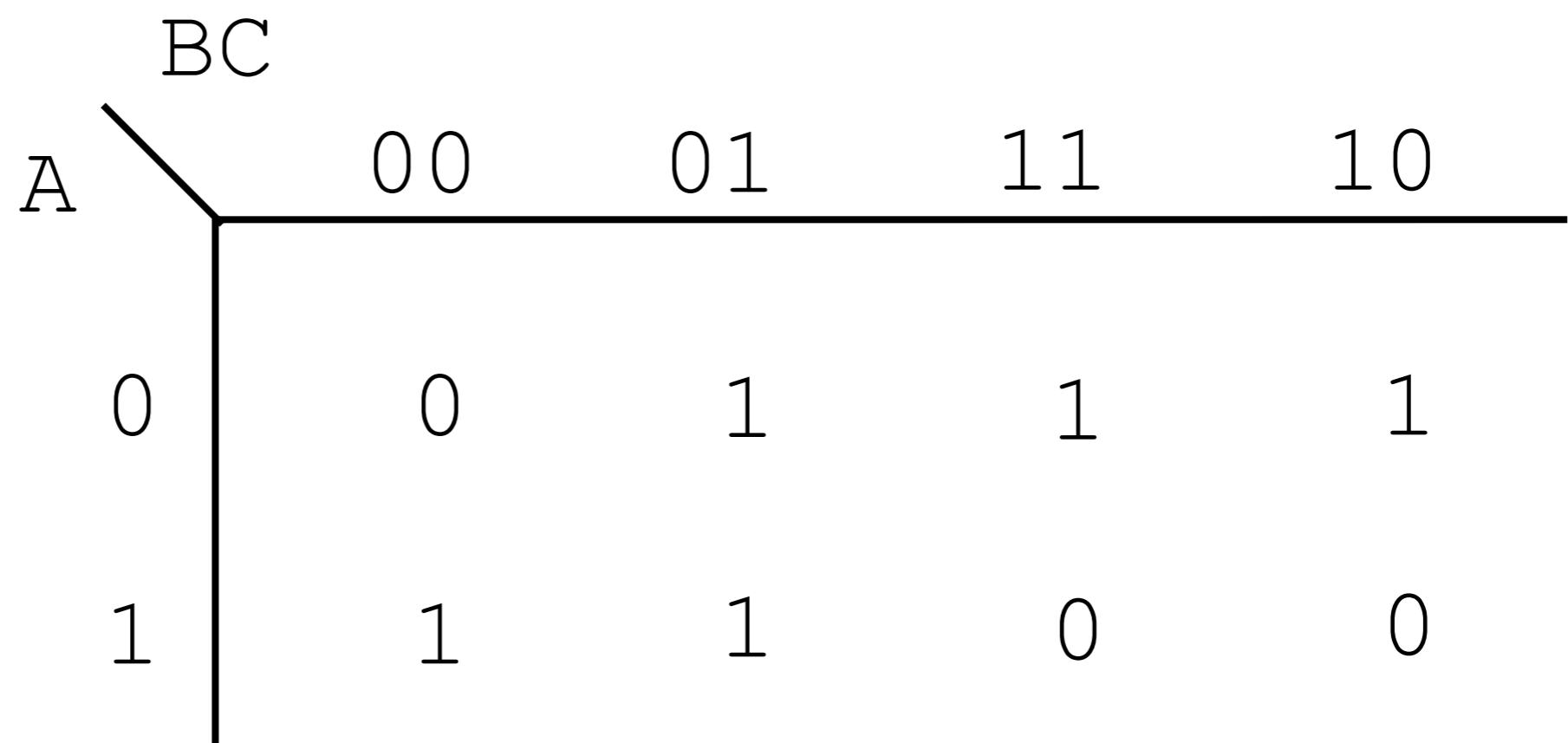
$$R = !A!BC + A!B!C + !ABC + !AB!C + A!BC$$

A	B	C	R
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

# Revisiting Problem

$$R = !A!BC + A!B!C + !ABC + !AB!C + A!BC$$

A	B	C	R
0	0	0	0
0	0	1	1
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0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

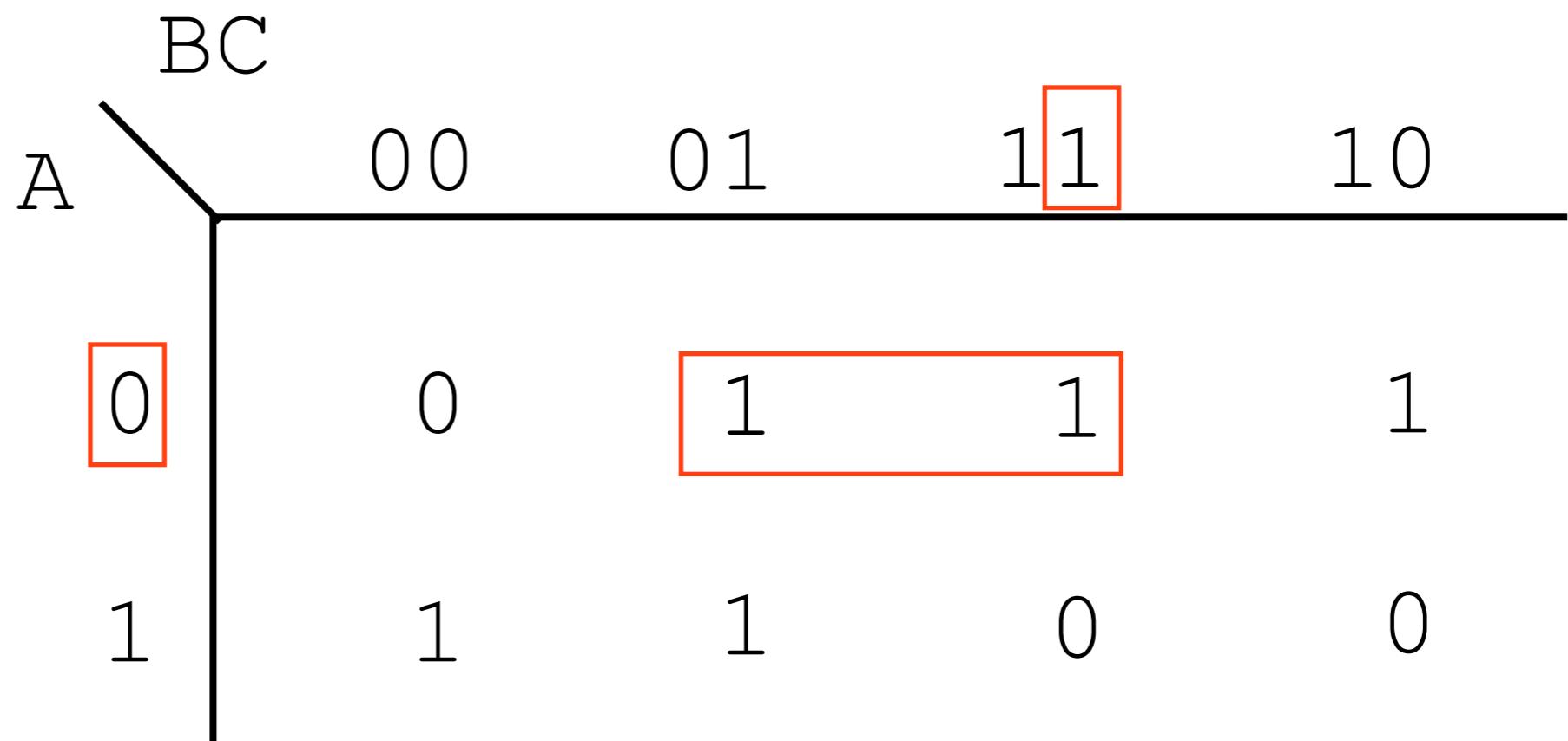


# Revisiting Problem

$$R = !A!BC + A!B!C + !ABC + !AB!C + A!BC$$

A	B	C	R
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

$$R = !AC$$

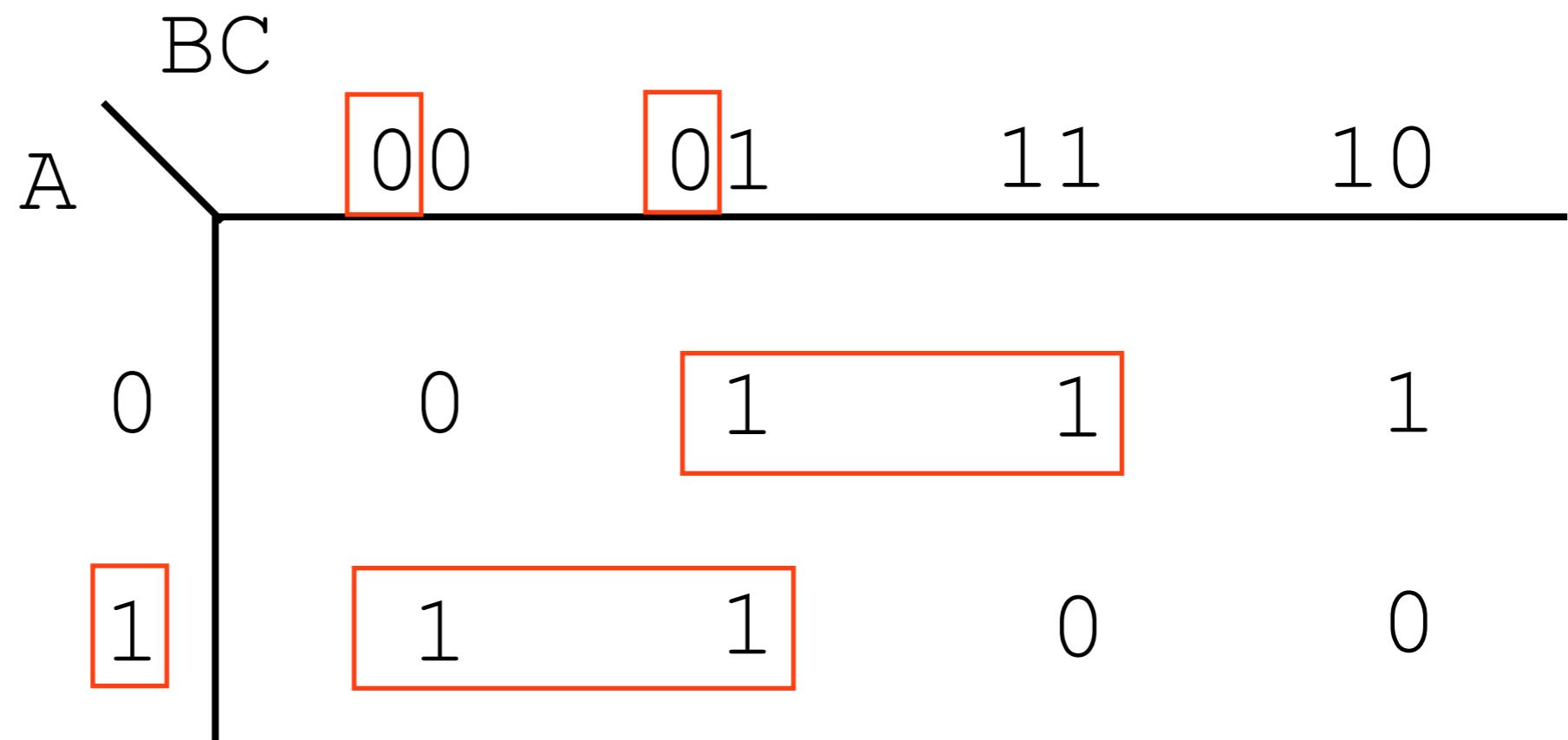


# Revisiting Problem

$$R = !A!BC + A!B!C + !ABC + !AB!C + A!BC$$

A	B	C	R
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

$$R = !AC + A!B$$

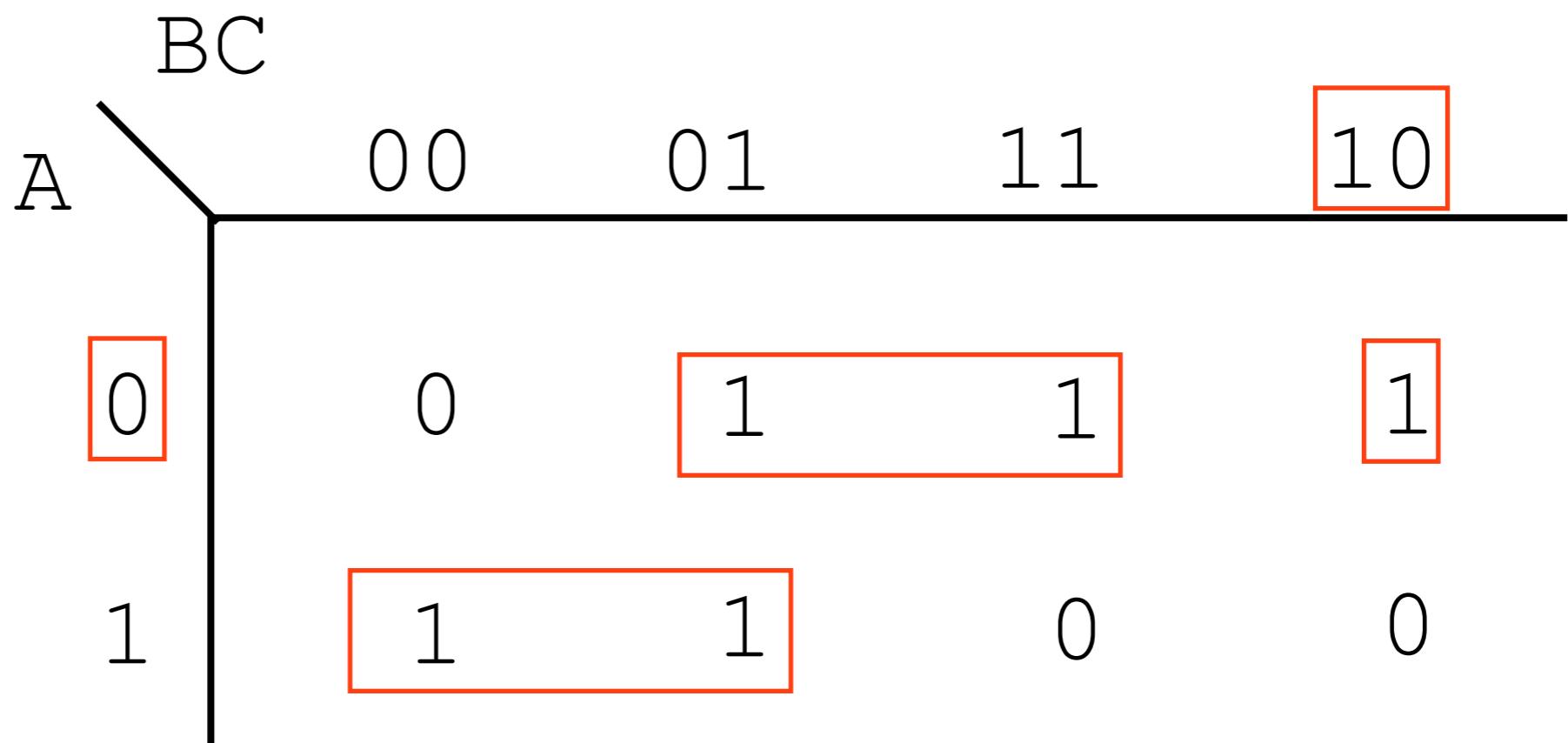


# Revisiting Problem

$$R = !A!BC + A!B!C + !ABC + !AB!C + A!BC$$

A	B	C	R
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

$$R = !AC + A!B + !AB!C$$



# Difference

- Algebraic solution:  $\bar{B}\bar{C} + \bar{A}\bar{B}\bar{C} + \bar{A}\bar{B}$
- K-map solution:  $\bar{A}\bar{C} + \bar{A}\bar{B} + \bar{A}\bar{B}\bar{C}$
- Question: why might these differ?

# Difference

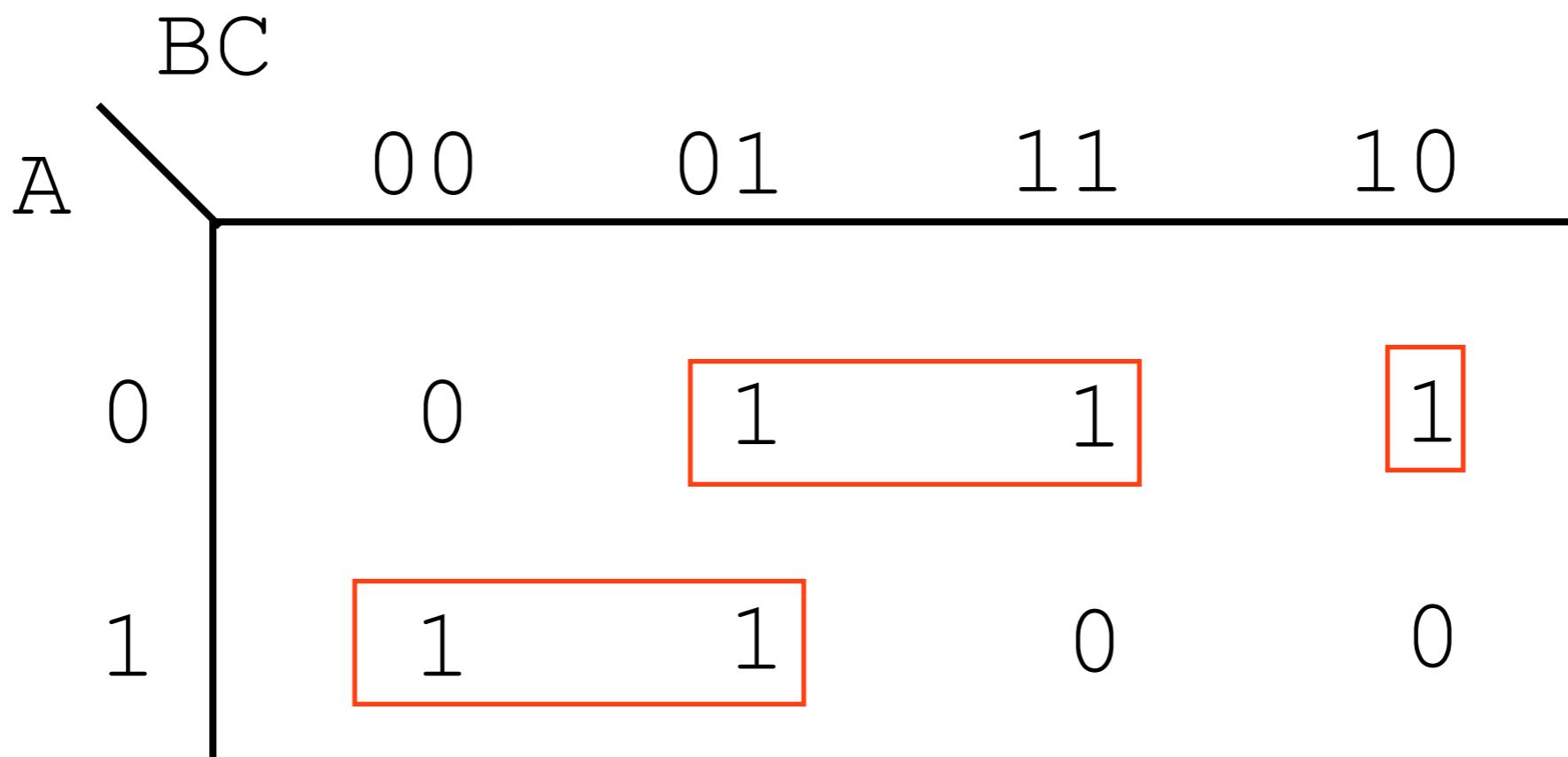
- Algebraic solution:  $\neg BC + A\neg B\neg C + \neg AB$
- K-map solution:  $\neg AC + A\neg B + \neg AB\neg C$
- Question: why might these differ?
  - Both are *minimal*, in that they have the fewest number of products possible
  - Can be multiple minimal solutions

# Difference

- Algebraic solution:  $\neg BC + A\neg B\neg C + \neg AB$
- K-map solution:  $\neg AC + A\neg B + \neg AB\neg C$
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Algebraic solution:  $\neg BC + \neg A \neg B \neg C + \neg A \neg B$   
K-map solution:  $\neg AC + \neg A \neg B + \neg A \neg B \neg C$



# Difference

Algebraic solution:  $\text{!BC} + \text{A}\text{!B}\text{!C} + \text{!AB}$   
K-map solution:  $\text{!BC} + \text{A}\text{!B}\text{!C} + \text{!AB}$

