CS64 Week 5 Lecture 2

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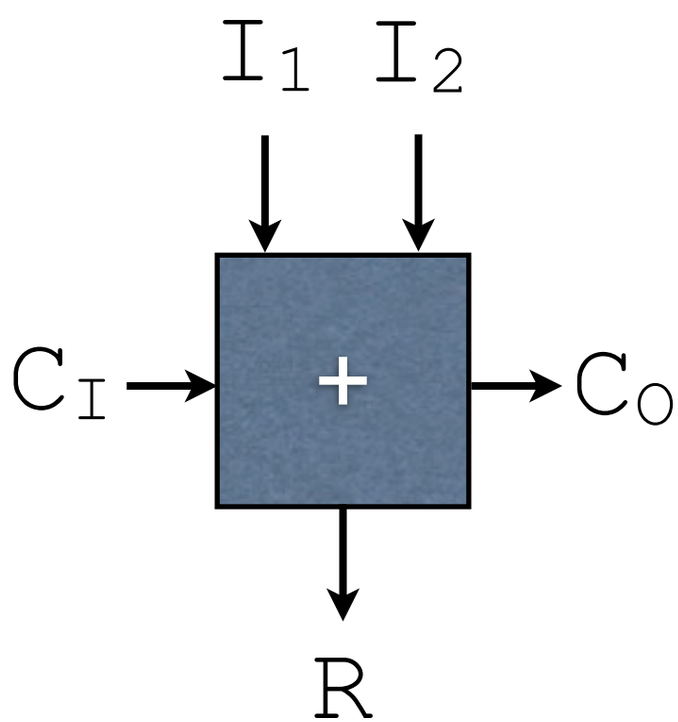
Overview

- More introduction to circuits
- Digital design: single bit adders
- Circuit minimization
  - Boolean algebra
  - Karnaugh maps
  - Exploiting *don't cares*

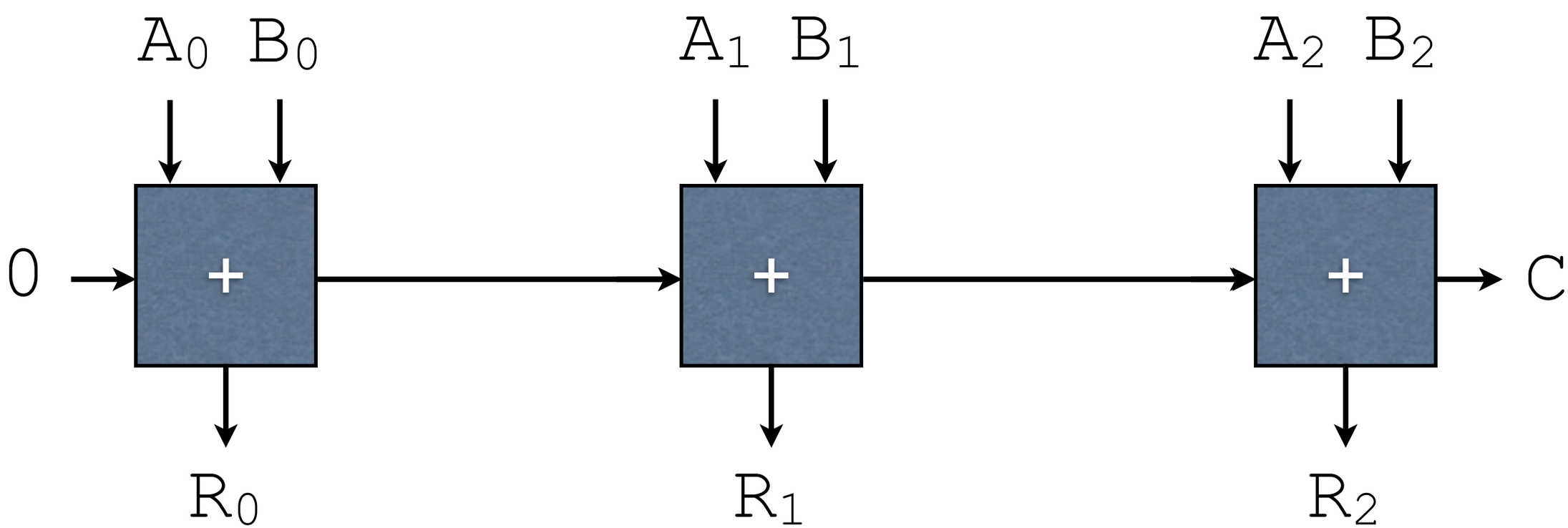
More Introduction to Circuits

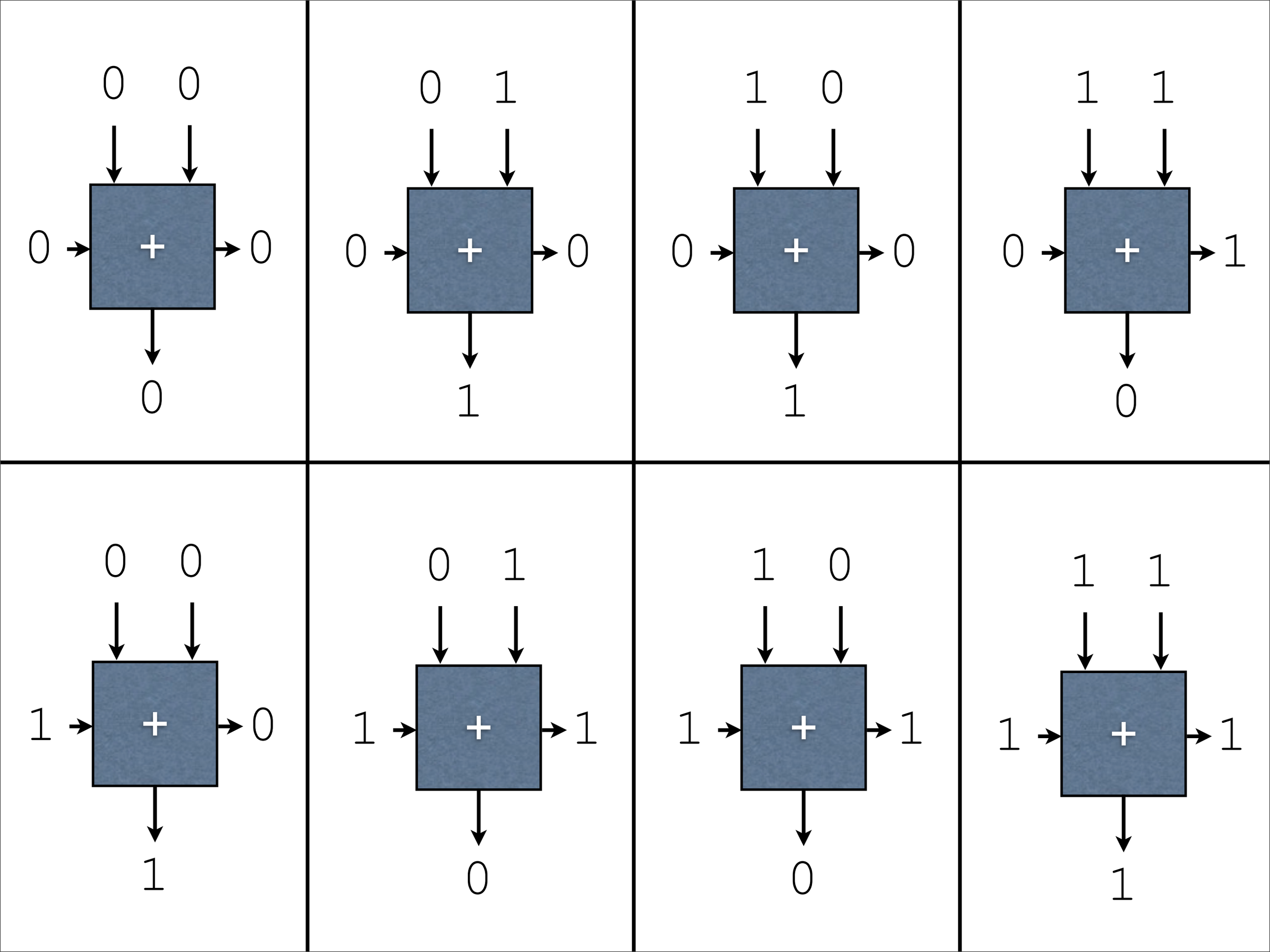
Recall: Single Bit Adders



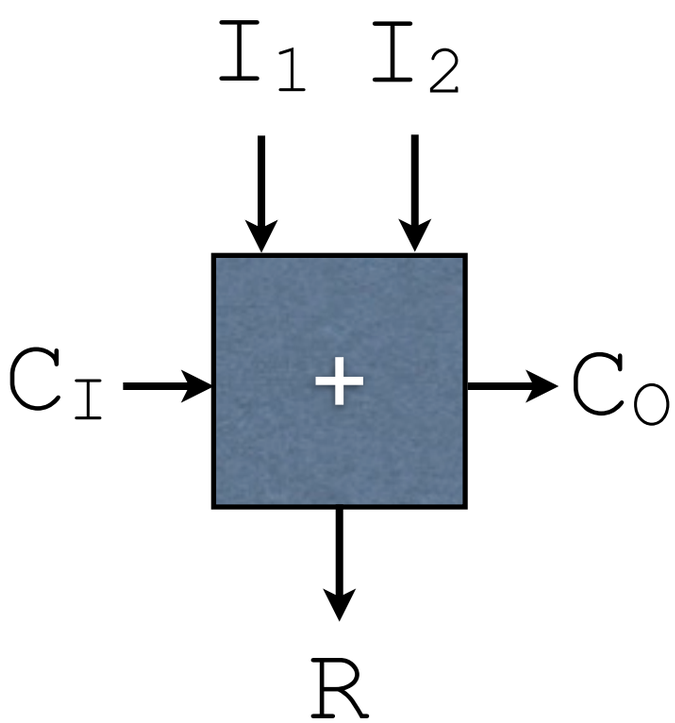
Stringing them Together

For two three-bit numbers, `A` and `B`, resulting in a three-bit result R




0 0
0 + 0
0
0 1
0 + 0
1
1 0
0 + 0
1
1 1
0 + 1
0
0 0
1 + 0
1
0 1
1 + 1
0
1 0
1 + 1
0
1 1
1 + 1
1

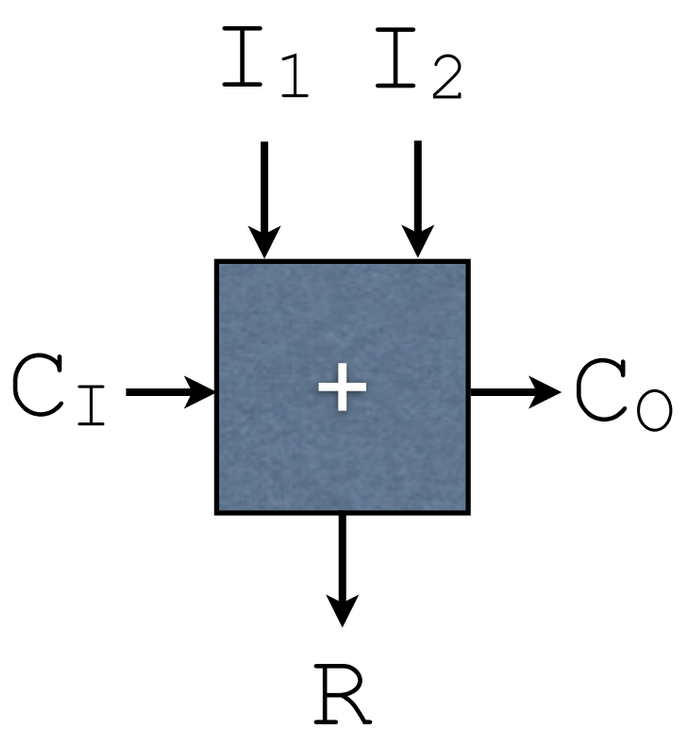
As a Truth Table



C_I	I_1	I_2	C_O	R
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

As a Truth Table

Question: how can this be turned into a circuit?



C_I	I_1	I_2	C_O	R
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

Sum of Products

- Variables: A, B, C...
- Negation of a variable: $\overline{A}$, $\overline{B}$, $\overline{C}$...

Sum of Products

- Another way to look at OR: sum (+)

```
A + B
```

- Another way to look at AND: multiplication (*)

```
A * B                    AB
```

Sum of Products Example

A	B	O
0	0	0
0	1	1
1	0	1
1	1	0

Sum of Products Example

A	B	O
0	0	0
0	1	1
1	0	1
1	1	0

Sum of Products Example

A	B	O
0	0	0
0	1	1
1	0	1
1	1	0

$$O = \boxed{\overline{A} * B}$$

Sum of Products Example

A	B	O
0	0	0
0	1	1
1	0	1
1	1	0

$$O = \overline{A}*B + \boxed{A*\overline{B}}$$

Sum of Products

C_I	I_1	I_2	C_O	R
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

Question: What would the sum of products look like for this table?

(Note: need one equation for each output.)

Sum of Products

C_I	I_1	I_2	C_O	R
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

Question: What would the sum of products look like for this table?

(Note: need one equation for each output.)

Answer in the presenter notes.

In-Class Example: Shift Left by 1

Circuit Minimization

Motivation

- Unnecessarily large programs: bad
- Unnecessarily large circuits: Very Bad™
  - Why?

Motivation

- Unnecessarily large programs: bad
- Unnecessarily large circuits: Very Bad™
  - Why?
    - Bigger circuits = bigger chips = higher cost (non-linear too!)
    - Longer circuits = more time needed to move electrons through = slower

Simplification

- Real-world formulas can often be simplified, according to algebraic rules
  - How might we simplify the following?

```
R = A*B + !A*B
```

Simplification

- Real-world formulas can often be simplified, according to algebraic rules
  - How might we simplify the following?

```
R = A*B + !A*B

R = B(A + !A)

R = B(true)

R = B
```

Simplification Trick

- Look for products that differ only in one variable
  - One product has the original variable (`A`)
  - The other product has the other variable (`!A`)

```
R = A*B + !A*B
```

Additional Example 1

!ABCD + ABCD + !AB!CD + AB!CD

Additional Example 1

$$!ABCD + ABCD + !AB!CD + AB!CD$$

$$BCD(A + !A) + !AB!CD + AB!CD$$

Additional Example 1

!ABCD + ABCD + !AB!CD + AB!CD

BCD(A + !A) + !AB!CD + AB!CD

BCD + !AB!CD + AB!CD

Additional Example 1

```
!ABCD + ABCD + !AB!CD + AB!CD

BCD(A + !A) + !AB!CD + AB!CD

BCD + !AB!CD + AB!CD

BCD + B!CD(!A + A)
```

Additional Example 1

```
!ABCD + ABCD + !AB!CD + AB!CD

BCD(A + !A) + !AB!CD + AB!CD

BCD + !AB!CD + AB!CD

BCD + B!CD(!A + A)

BCD + B!CD
```

Additional Example 1

```
!ABCD + ABCD + !AB!CD + AB!CD

BCD(A + !A) + !AB!CD + AB!CD

BCD + !AB!CD + AB!CD

BCD + B!CD(!A + A)

BCD + B!CD

BD(C + !C)
```

Additional Example 1

```
!ABCD + ABCD + !AB!CD + AB!CD

BCD(A + !A) + !AB!CD + AB!CD

BCD + !AB!CD + AB!CD

BCD + B!CD(!A + A)

BCD + B!CD

BD(C + !C)
BD
```

Additional Example 2

`!A!BC + A!B!C + !ABC + !AB!C + A!BC`

Additional Example 2

!A!BC + A!B!C + !ABC + !AB!C + A!BC

!A!BC + A!BC + A!B!C + !ABC + !AB!C

Additional Example 2

!A!BC + A!B!C + !ABC + !AB!C + A!BC

!A!BC + A!BC + A!B!C + !ABC + !AB!C

!BC(A + !A) + A!B!C + !ABC + !AB!C

Additional Example 2

!A!BC + A!B!C + !ABC + !AB!C + A!BC

!A!BC + A!BC + A!B!C + !ABC + !AB!C

!BC(A + !A) + A!B!C + !ABC + !AB!C

!BC + A!B!C + !ABC + !AB!C

Additional Example 2

!A!BC + A!B!C + !ABC + !AB!C + A!BC

!A!BC + A!BC + A!B!C + !ABC + !AB!C

!BC(A + !A) + A!B!C + !ABC + !AB!C

!BC + A!B!C + !ABC + !AB!C

!BC + A!B!C + !AB(C + !C)

Additional Example 2

```
!A!BC + A!B!C + !ABC + !AB!C + A!BC

!A!BC + A!BC + A!B!C + !ABC + !AB!C

!BC(A + !A) + A!B!C + !ABC + !AB!C

!BC + A!B!C + !ABC + !AB!C

!BC + A!B!C + !AB(C + !C)

!BC + A!B!C + !AB
```

Scaling Up

- Performing this sort of algebraic manipulation by hand can be tricky
- We can use *Karnaugh maps* to make it immediately apparent as to what can be simplified

Example

```
R = A*B + !A*B
```

Example

```
R = A*B + !A*B
```

A	B	O
0	0	0
0	1	1
1	0	0
1	1	1

Example

R = A*B + !A*B

A	B	O
0	0	0
0	1	1
1	0	0
1	1	1

A \ B	0	1
0	0	1
1	0	1

Example

R = A*B + !A*B

A	B	O
0	0	0
0	1	1
1	0	0
1	1	1

A \ B	0	1
0	0	1
1	0	1

Example

R = A*B + !A*B

A	B	O
0	0	0
0	1	1
1	0	0
1	1	1

A \ B	0	1
0	0	1
1	0	1

Example

R = A*B + !A*B

A	B	O
0	0	0
0	1	1
1	0	0
1	1	1

R = B

A \ B	0	1
0	0	1
1	0	1

Three Variables

- We can scale this up to three variables, by combining two variables on one axis
- The combined axis must be arranged such that only one bit changes per position

A \ BC	00	01	11	10
0	?	?	?	?
1	?	?	?	?

Three Variable Example

$$R = !A!BC + !ABC + A!BC + ABC$$

R = !A!BC + !ABC + A!BC + ABC

A	B	C	R
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

$$R = !A!BC + !ABC + A!BC + ABC$$

A	B	C	R
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

A \ BC	00	01	11	10
0	0	1	1	0
1	0	1	1	0

$$R = !A!BC + !ABC + A!BC + ABC$$

A	B	C	R
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

A \ BC	00	01	11	10
0	0	1	1	0
1	0	1	1	0

$$R = !A!BC + !ABC + A!BC + ABC$$

A	B	C	R
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

$$R = C$$

A \ BC	00	01	11	10
0	0	1	1	0
1	0	1	1	0

Another Three Variable Example

$$R = !A!B!C + !A!BC + !ABC + !AB!C + A!B!C + AB!C$$

$$R = !A!B!C + !A!BC + !ABC + !AB!C + A!B!C + AB!C$$

A	B	C	R
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

$$R = !A!B!C + !A!BC + !ABC + !AB!C + A!B!C + AB!C$$

A	B	C	R
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

A \ BC	00	01	11	10
0	1	1	1	1
1	1	0	0	1

$$R = !A!B!C + !A!BC + !ABC + !AB!C + A!B!C + AB!C$$

A	B	C	R
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

A \ BC	00	01	11	10
0	1	1	1	1
1	1	0	0	1

$$R = !A!B!C + !A!BC + !ABC + !AB!C + A!B!C + AB!C$$

A	B	C	R
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

A \ BC	00	01	11	10
0	1	1	1	1
1	1	0	0	1

$$R = !A!B!C + !A!BC + !ABC + !AB!C + A!B!C + AB!C$$

A	B	C	R
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

A \ BC	00	01	11	10
0	1	1	1	1
1	1	0	0	1

$$R = !A!B!C + !A!BC + !ABC + !AB!C + A!B!C + AB!C$$

A	B	C	R
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

$$R = !A + !C$$

A \ BC	00	01	11	10
0	1	1	1	1
1	1	0	0	1

Four Variable Example

$$R = !A!B!C!D + !A!B!CD + !A!BC!D + !ABC!D + A!B!C!D + A!B!CD + A!BC!D$$

R = !A!B!C!D + !A!B!CD + !A!BC!D + !ABC!D + A!B!C!D + A!B!CD + A!BC!D

AB \ CD	00	01	11	10
00	1	1	0	1
01	0	0	0	1
11	0	0	0	0
10	1	1	0	1

R = !A!B!C!D + !A!B!CD + !A!BC!D + !ABC!D + A!B!C!D + A!B!CD + A!BC!D

AB \ CD	00	01	11	10
00	1	1	0	1
01	0	0	0	1
11	0	0	0	0
10	1	1	0	1

R = !A!B!C!D + !A!B!CD + !A!BC!D + !ABC!D + A!B!C!D + A!B!CD + A!BC!D

R =!B!C

AB \ CD	00	01	11	10
00	1	1	0	1
01	0	0	0	1
11	0	0	0	0
10	1	1	0	1

R = !A!B!C!D + !A!B!CD + !A!BC!D + !ABC!D + A!B!C!D + A!B!CD + A!BC!D

R =!B!C + !B!D

AB \ CD	00	01	11	10
00	1	1	0	1
01	0	0	0	1
11	0	0	0	0
10	1	1	0	1

R = !A!B!C!D + !A!B!CD + !A!BC!D + !ABC!D + A!B!C!D + A!B!CD + A!BC!D

R =!B!C + !B!D + !AC!D

AB \ CD	00	01	11	10
00	1	1	0	1
01	0	0	0	1
11	0	0	0	0
10	1	1	0	1

K-Map Rules in Summary (1)

- Groups can contain only `1`s
- Only `1`s in adjacent groups are allowed (no diagonals)
- The number of `1`s in a group must be a power of two (1, 2, 4, 8...)
- The groups must be as large as legally possible

K-Map Rules in Summary (2)

- All `1`s must belong to a group, even if it's a group of one element
- Overlapping groups are permitted
- Wrapping around the map is permitted
- Use the fewest number of groups possible

Revisiting Problem

!A!BC + A!B!C + !ABC + !AB!C + A!BC

Revisiting Problem

R = !A!BC + A!B!C + !ABC + !AB!C + A!BC

A	B	C	R
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

Revisiting Problem

R = !A!BC + A!B!C + !ABC + !AB!C + A!BC

A	B	C	R
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

A \ BC	00	01	11	10
0	0	1	1	1
1	1	1	0	0

Revisiting Problem

R = !A!BC + A!B!C + !ABC + !AB!C + A!BC

A	B	C	R
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

R = !AC

A \ BC	00	01	11	10
0	0	1	1	1
1	1	1	0	0

Revisiting Problem

R = !A!BC + A!B!C + !ABC + !AB!C + A!BC

A	B	C	R
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

R = !AC + A!B

A \ BC	00	01	11	10
0	0	1	1	1
1	1	1	0	0

Revisiting Problem

R = !A!BC + A!B!C + !ABC + !AB!C + A!BC

A	B	C	R
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

R = !AC + A!B + !AB!C

A \ BC	00	01	11	10
0	0	1	1	1
1	1	1	0	0

Difference

- Algebraic solution: `!BC + A!B!C + !AB`
- K-map solution: `!AC + A!B + !AB!C`
- Question: why might these differ?

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- K-map solution: `!AC + A!B + !AB!C`
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  - Both are *minimal*, in that they have the fewest number of products possible
  - Can be multiple minimal solutions

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- Algebraic solution: `!BC + A!B!C + !AB`
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Difference

Algebraic solution: `!BC + A!B!C + !AB`

K-map solution: `!AC + A!B + !AB!C`

A \ BC	00	01	11	10
0	0	1	1	1
1	1	1	0	0

Difference

Algebraic solution: !BC + A!B!C + !AB
K-map solution: !BC + A!B!C + !AB

A \ BC	00	01	11	10
0	0	1	1	1
1	1	1	0	0

Exploiting *Don't Cares* in K-Maps

Don't Cares

- Occasionally, a circuit's output will be unspecified on a given input
  - Occurs when an input's value is invalid
- In these situations, we say the output is a *don't care*, marked as an `X` in a truth table

Example: Binary Coded Decimal

- Occasionally, it is convenient to represent decimal numbers directly in binary, using 4-bits per decimal digit
  - For example, a digital display

Example: Binary Coded Decimal

- Not all binary values map to decimal digits

Binary	Decimal
0000	0
0001	1
0010	2
0011	3
0100	4
0101	5
0110	6
0111	7

Binary	Decimal
1000	8
1001	9
1010	X
1011	X
1100	X
1101	X
1110	X
1111	X

Significance

- Recall that in a K-map, we can only group `1`s
- Because the value of a *don't care* is irrelevant, we can treat it as a `1` if it is convenient to do so (or a `0` if that would be more convenient)

Example

- A circuit that calculates if the binary coded decimal input `% 2 == 0`

Example

- A circuit that calculates if the binary coded decimal input `% 2 == 0`

I_3	I_2	I_1	I_0	R
0	0	0	0	1
0	0	0	1	0
0	0	1	0	1
0	0	1	1	0
0	1	0	0	1
0	1	0	1	0
0	1	1	0	1
0	1	1	1	0

I_3	I_2	I_1	I_0	R
1	0	0	0	1
1	0	0	1	0
1	0	1	0	X
1	0	1	1	X
1	1	0	0	X
1	1	0	1	X
1	1	1	0	X
1	1	1	1	X

Example

As a K-map

I_3I_2 \ I_1I_0	00	01	11	10
00	1	0	1	0
01	1	0	1	0
11	X	X	X	X
10	1	0	X	X

Example

If we don't exploit *don't cares*...

I_3I_2 \ I_1I_0	00	01	11	10
00	1	0	1	0
01	1	0	1	0
11	X	X	X	X
10	1	0	X	X

Example

If we **do** exploit *don't cares*...

I_3I_2 \ I_1I_0	00	01	11	10
00	1	0	1	0
01	1	0	1	0
11	X	X	X	X
10	1	0	X	X

Example

If we **do** exploit *don't cares*...

$$R = !I_1!I_0 + I_1I_0$$

I_3I_2 \ I_1I_0	00	01	11	10
00	1	0	1	0
01	1	0	1	0
11	X	X	X	X
10	1	0	X	X