CS 64 Week 1 Lecture 2

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Overview

- Wrapping up working with different bases
- Bitwise operations
- Two's complement
- Addition
- Subtraction
- Multiplication (if time)

Wrapping Up Working with Different Bases

Hexadecimal

- Base 16
- Binary is horribly inconvenient to write out
- Easier to convert between hexadecimal (which is more convenient) and binary
  - Each hexadecimal digit maps to four binary digits
  - Can just memorize a table

Hexadecimal

- Digits 0-9, along with A (10), B (11), C (12), D (13), E (14), F (15)

Hexadecimal Example

- What is 1AF hexadecimal in decimal?

Hexadecimal Example

I	A	F

Hexadecimal Example

1	A	F
Two-fifty-sixes	Sixteens	Ones

Hexadecimal Example

1	A	F
Two-fifty-sixes 1×16^2	Sixteens 10×16^1	Ones 15×16^0

Hexadecimal Example

1	A	F
Two-fifty-sixes	Sixteens	Ones
1×16^2	10×16^1	15×16^0
	16 16 16 16 16 16 16 16 16 16	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
256	(160)	(15)

Hexadecimal to Binary

- Previous techniques all work, using decimal as an intermediate
- The faster way: memorize a table (which can be easily reconstructed)

Hexadecimal to Binary

Hexadecimal	Binary
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111

Hexadecimal	Binary
8	1000
9	1001
A (10)	1010
B (11)	1011
C (12)	1100
D (13)	1101
E (14)	1110
F (15)	1111

Bitwise Operations

Bitwise AND

- Similar to logical AND (`&&`), except it works on a bit-by-bit manner
- Denoted by a single ampersand: `&`

```
(1001 &
 0101)=
 0001
```

Bitwise OR

- Similar to logical OR (||), except it works on a bit-by-bit manner
- Denoted by a single pipe character: |

```
(1001 |
 0101) =
 1101
```

Bitwise XOR

- Exclusive OR, denoted by a carat: ^
- Similar to bitwise OR, except that if both inputs are `1` then the result is `0`

```
(1001 ^
 0101)=
 1100
```

Bitwise NOT

- Similar to logical NOT (`!`), except it works on a bit-by-bit manner
- Denoted by a tilde character: `~`

```
~1001 =
 0110
```

Shift Left

- Move all the bits `N` positions to the left, subbing in `N` `0`s on the right

Shift Left

- Move all the bits `N` positions to the left, subbing in `N` `0`s on the right

`1001`

Shift Left

- Move all the bits `N` positions to the left, subbing in `N` `0`s on the right

```
1001 << 2 =
100100
```

Shift Left

- Useful as a restricted form of multiplication
- Question: how?

```
1001 << 2 =
100100
```

Shift Left as Multiplication

- Equivalent decimal operation:

```
234
```

Shift Left as Multiplication

- Equivalent decimal operation:

```
234 << 1 =
2340
```

Shift Left as Multiplication

- Equivalent decimal operation:

```
234 << 1 =
2340

234 << 2 =
23400
```

Multiplication

- Shifting left `N` positions multiplies by `(base)`$^{\texttt{N}}$
- Multiplying by 2 or 4 is often necessary (shift left 1 or 2 positions, respectively)
- Often a whooole lot faster than telling the processor to multiply
- Compilers try hard to do this

```
234 << 2 =
23400
```

Shift Right

- Move all the bits `N` positions to the right, subbing in **either** `N` `0`s or `N` `1`s on the left
  - Two different forms

Shift Right

- Move all the bits `N` positions to the right, subbing in **either** `N` `0`s or `N` (whatever the leftmost bit is)s on the left
  - Two different forms

```
1001 >> 2 =
either 0010 or 1110
```

Shift Right Trick

- Question: If shifting left multiplies, what does shift right do?

Shift Right Trick

- Question: If shifting left multiplies, what does shift right do?
  - Answer: divides in a similar way, but truncates result

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Shift Right Trick

- Question: If shifting left multiplies, what does shift right do?
  - Answer: divides in a similar way, but truncates result

```
234 >> 1 =
23
```

Two Forms of Shift Right

- Subbing in `0`s makes sense
- What about subbing in the leftmost bit?
  - And why is this called "arithmetic" shift right?

```
1100 (arithmetic)>> 1 =
1110
```

Answer...Sort of

- Arithmetic form is intended for numbers in *twos complement*, whereas the non-arithmetic form is intended for *unsigned* numbers

Twos Complement

Problem

- Binary representation so far makes it easy to represent positive numbers and zero
- Question: What about representing negative numbers?

Twos Complement

- Way to represent positive integers, negative integers, and zero
- If `1` is in the *most significant bit* (generally leftmost bit in this class), then it is negative

Decimal to Twos Complement

- Example: -5 decimal to binary (twos complement)

Decimal to Twos Complement

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- First, convert the magnitude to an unsigned representation

Decimal to Twos Complement

- Example: -5 decimal to binary (twos complement)
- First, convert the magnitude to an unsigned representation

`5` (decimal) = `0101` (binary)

Decimal to Twos Complement

- Then, take the bits, and negate them

Decimal to Twos Complement

- Then, take the bits, and negate them

`0101`

Decimal to Twos Complement

- Then, take the bits, and negate them

```
~0101 =
 1010
```

Decimal to Twos Complement

- Finally, add one:

Decimal to Twos Complement

- Finally, add one:

```
1010
```

Decimal to Twos Complement

- Finally, add one:

```
1010 + 1 =
1011
```

Twos Complement to Decimal

- Same operation: negate the bits, and add one

Twos Complement to Decimal

- Same operation: negate the bits, and add one

```
1011
```

Twos Complement to Decimal

- Same operation: negate the bits, and add one

```
~1011 =
 0100
```

Twos Complement to Decimal

- Same operation: negate the bits, and add one

`0100`

Twos Complement to Decimal

- Same operation: negate the bits, and add one

```
0100 + 1 =
0101
```

Twos Complement to Decimal

- Same operation: negate the bits, and add one

```
0100 + 1 =
0101 =
-5
```

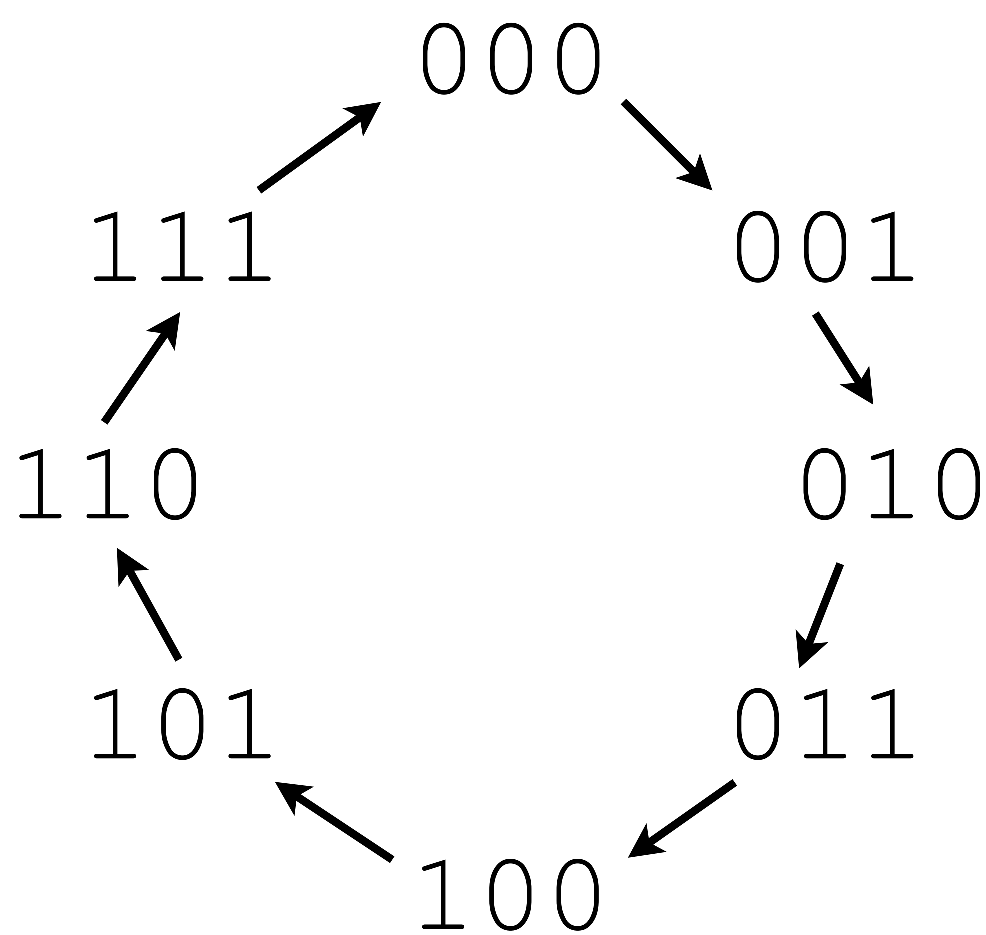
We started with `1011` - negative

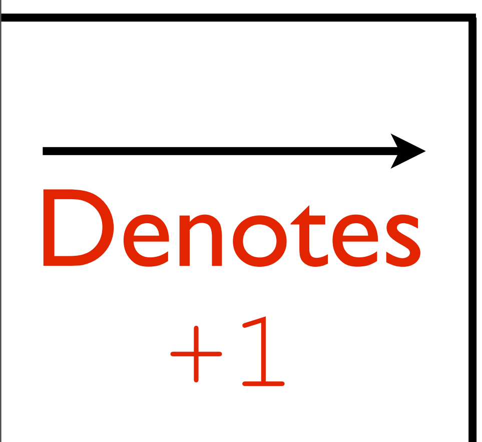
Where Is Twos Complement From?

- Intuition: try to subtract 1 from 0, in decimal
  - Involves borrowing from an invisible number on the left
  - Twos complement is based on the same idea

Another View

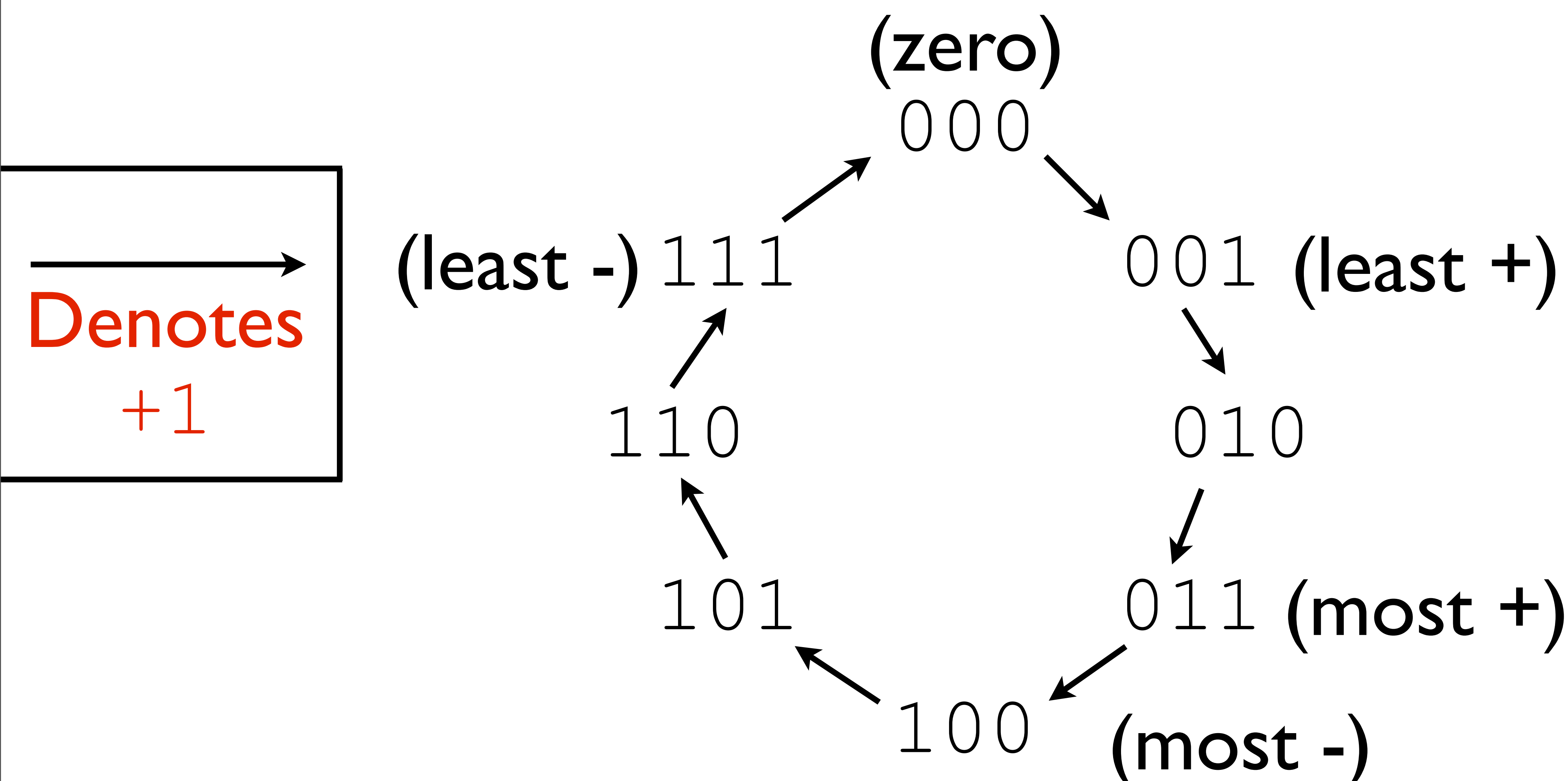
- Modular arithmetic, with the convention that a leading `1` bit means negative





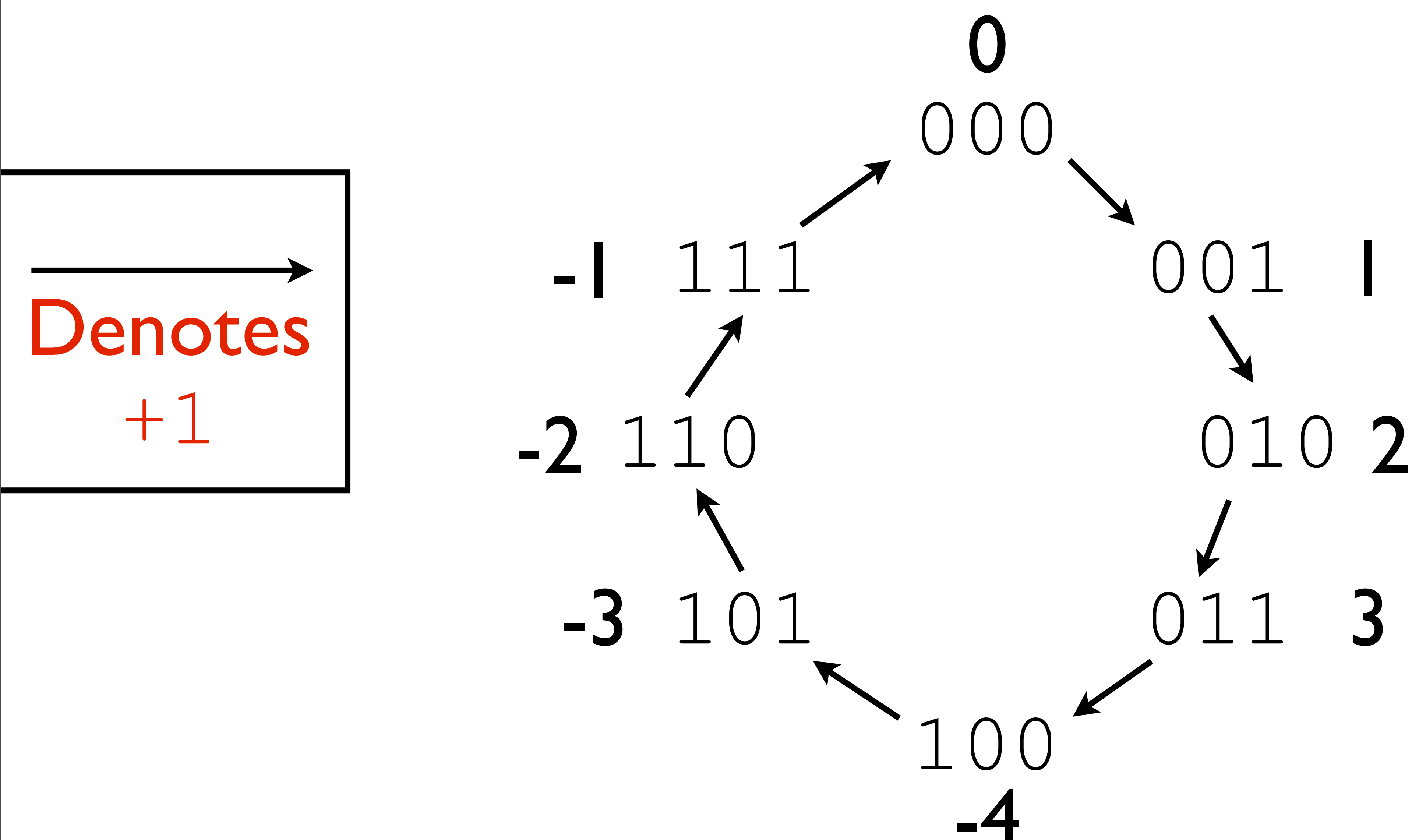
Another View

- Modular arithmetic, with the convention that a leading `1` bit means negative

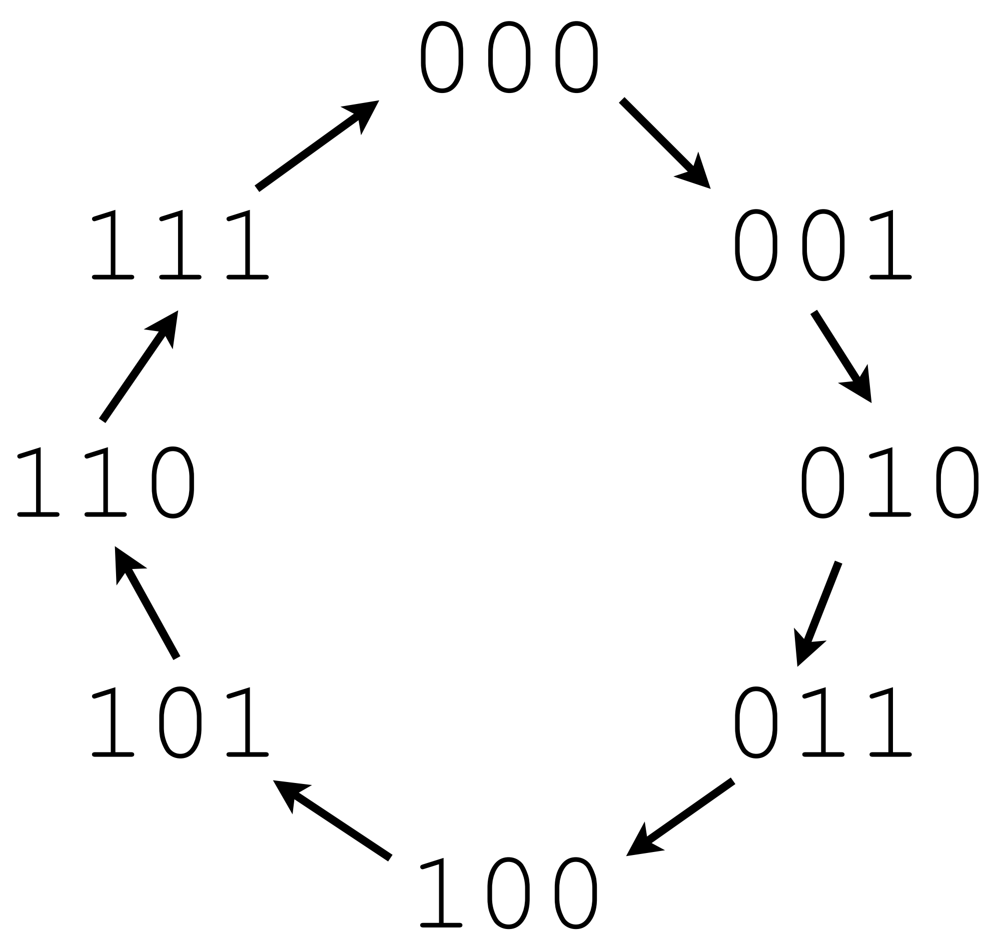


Another View

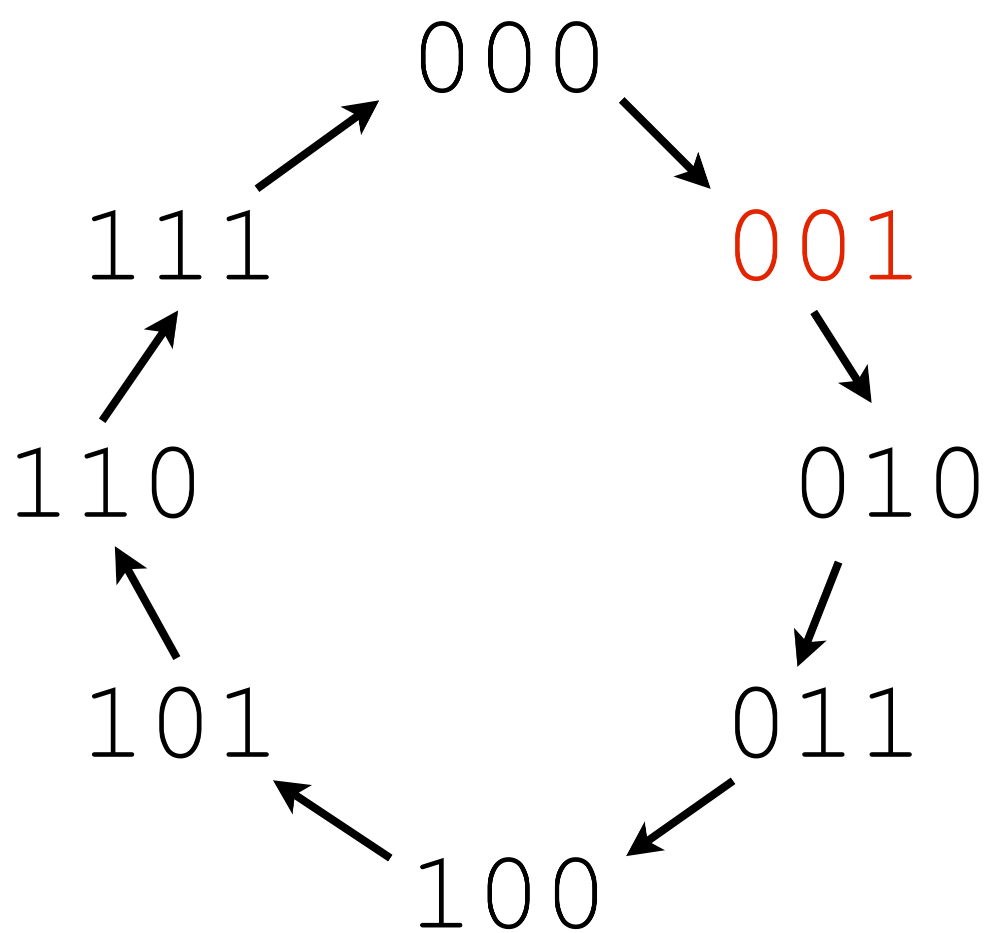
- Modular arithmetic, with the convention that a leading 1 bit means negative



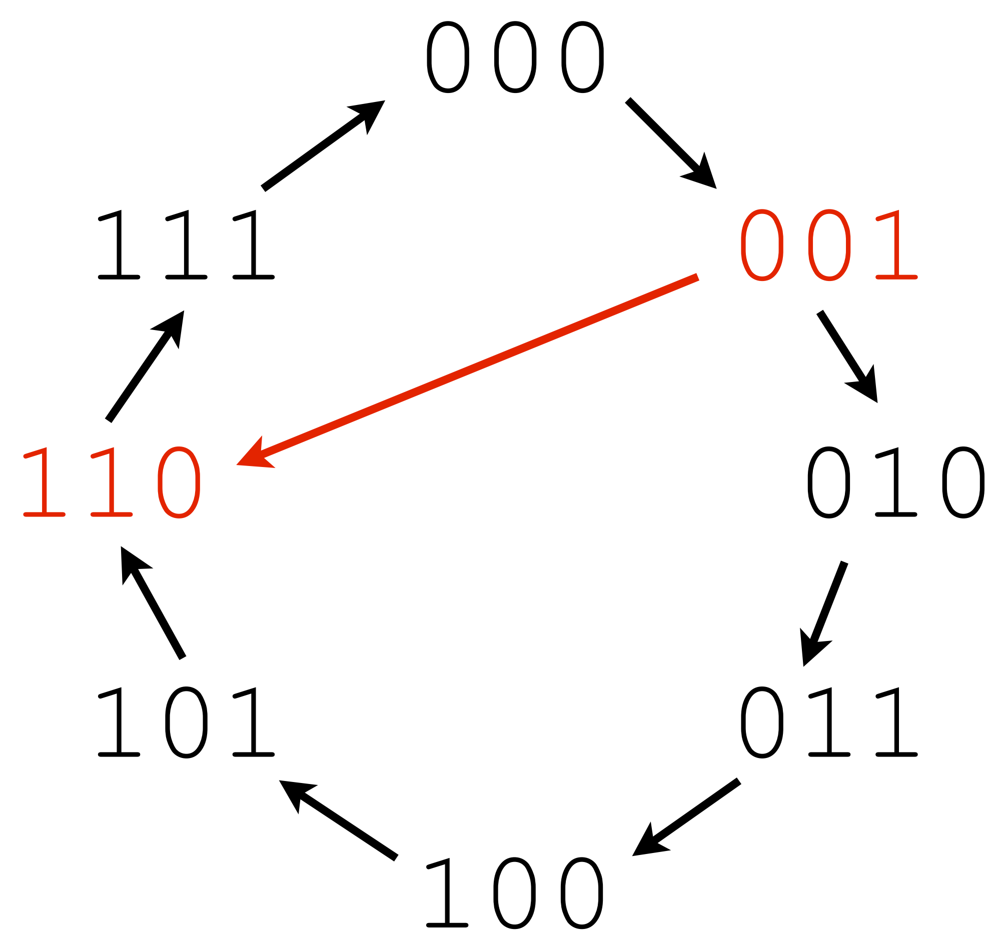
Negation of `1`



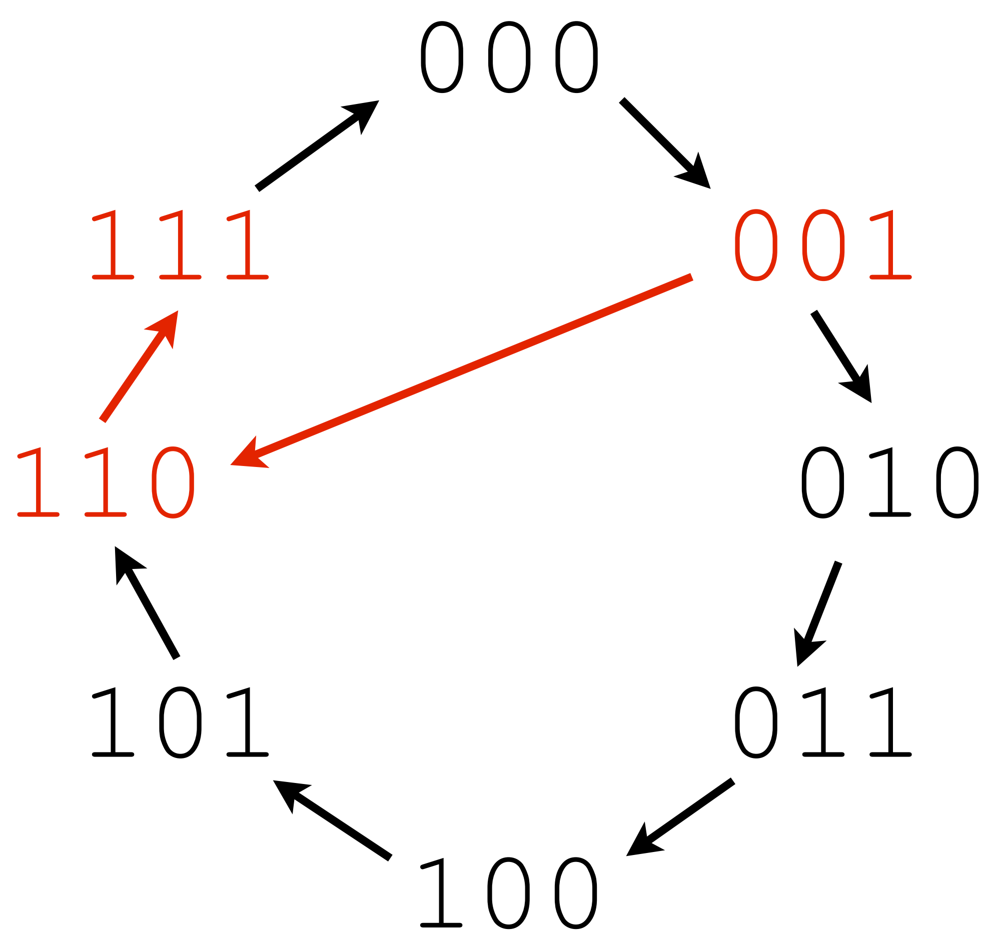
Negation of 1



Negation of `1`

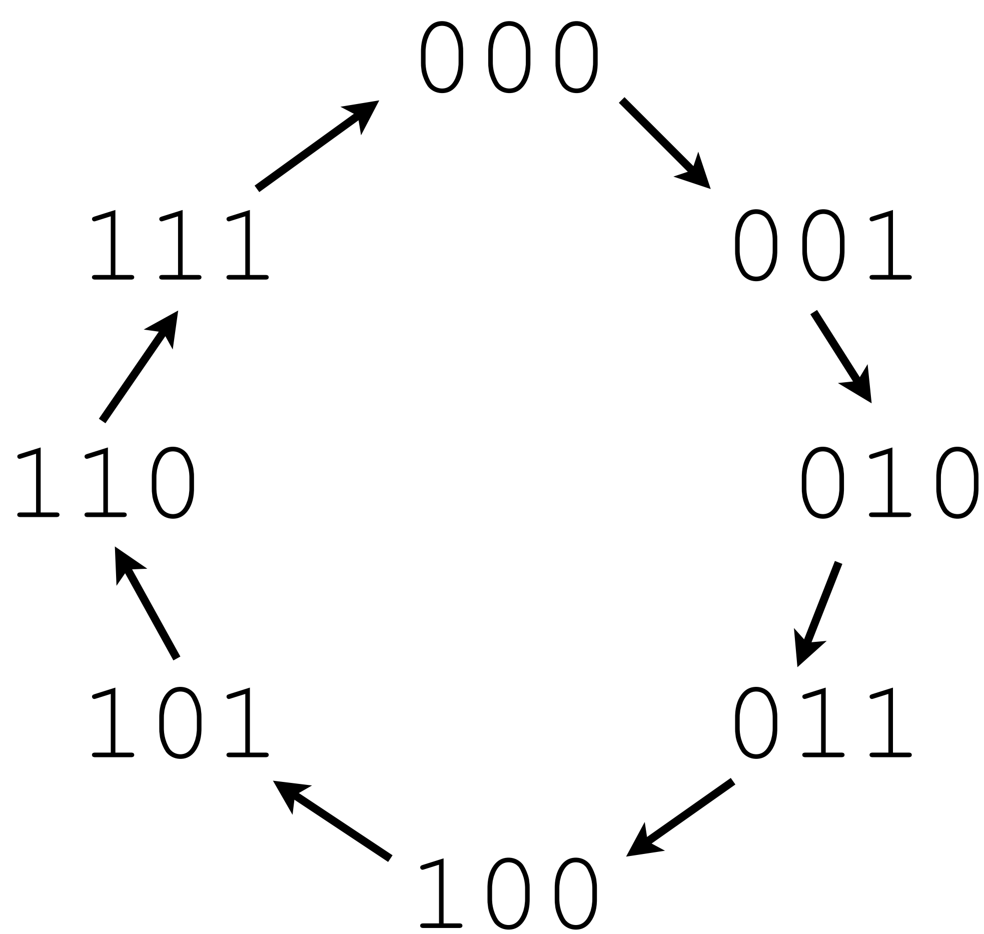


Negation of `1`



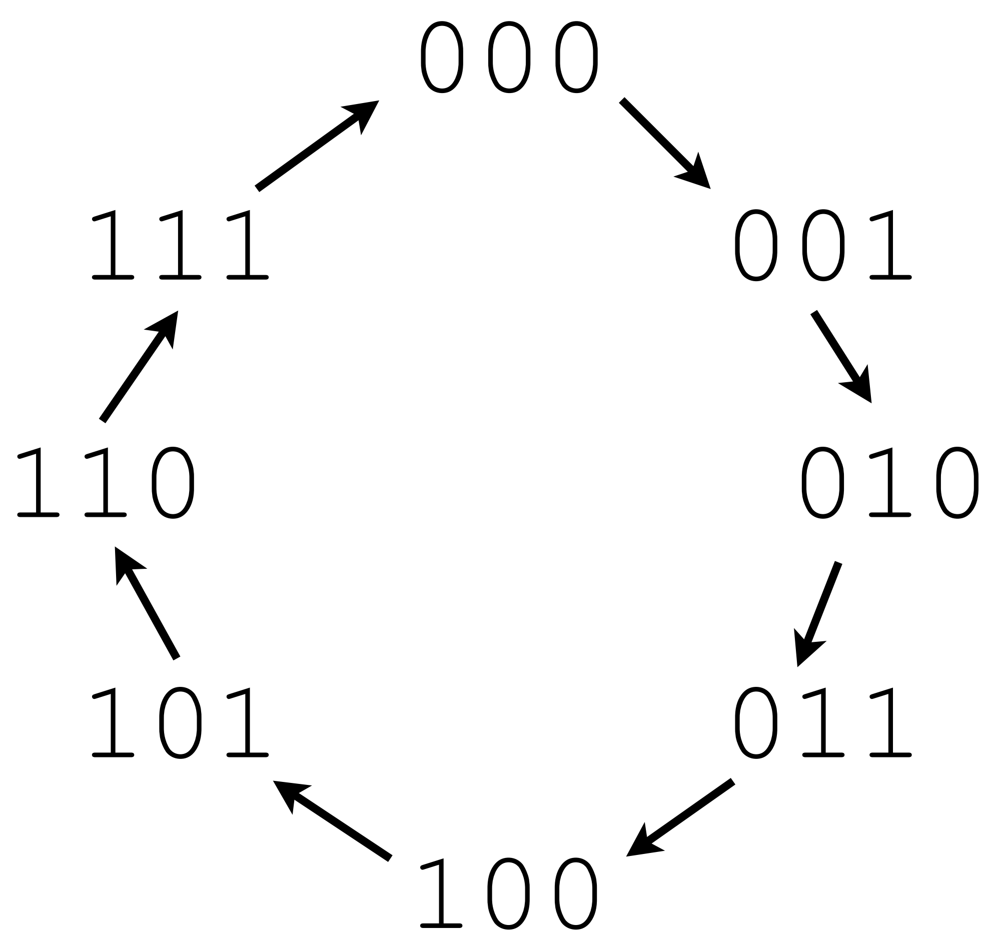
Consequences

- What is the negation of `000`?



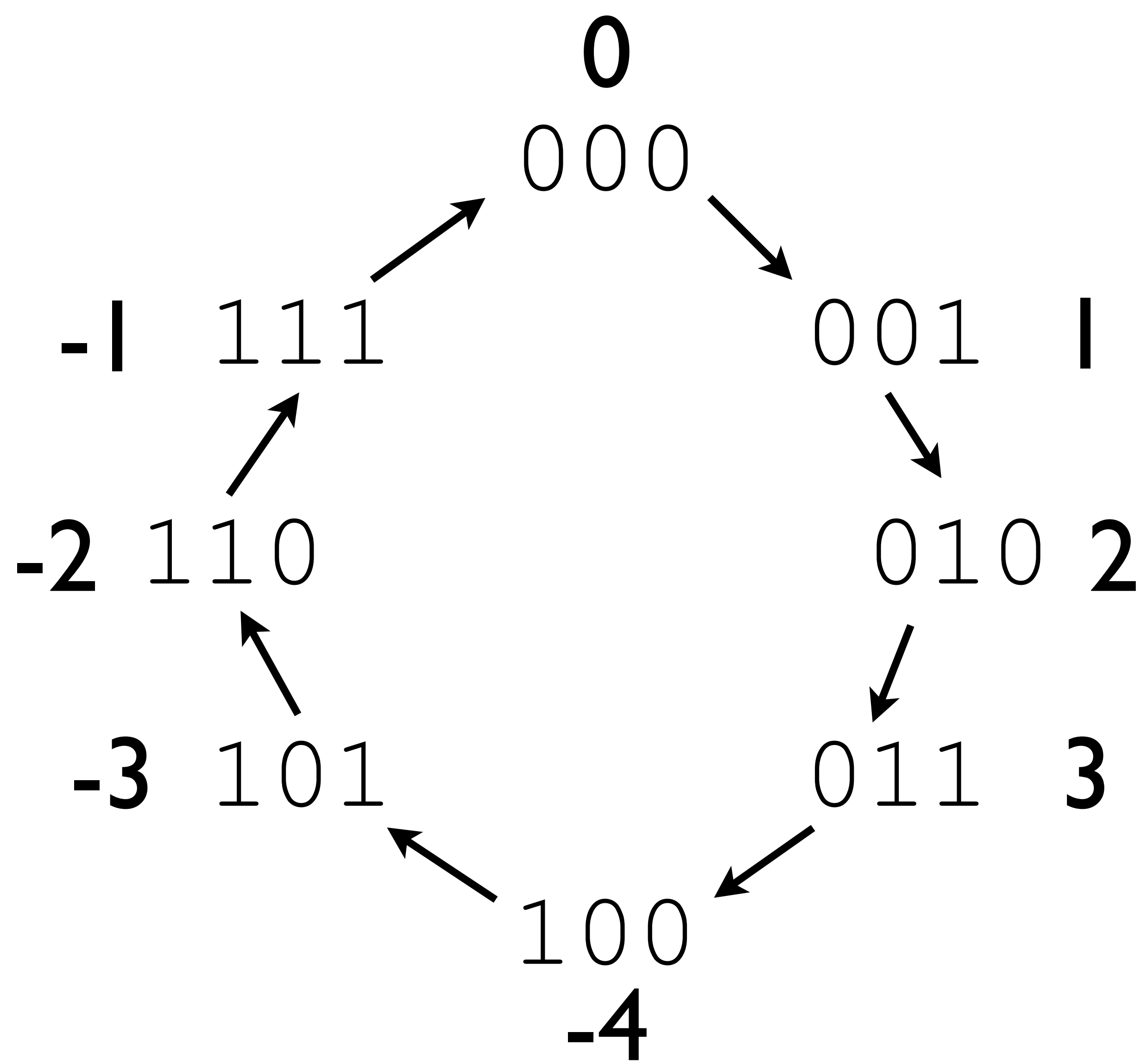
Consequences

- What is the negation of `100`?



Arithmetic Shift Right

- **Not exactly** division by a power of two
- Consider -3 / 2



Addition

Building Up Addition

- Question: how might we add the following, in decimal?

```
 986
+123
----
   ?
```

Building Up Addition

- Question: how might we add the following, in decimal?

```
 986
+123
----
   ?
```

			`6`
			`+3`
			`--`
			`?`

Building Up Addition

- Question: how might we add the following, in decimal?

```
 986
+123
----
   ?
```

		` 8`
		`+2`
		`--`
		` ?`
			` 6`
			`+3`
			`--`
			` 9`

Building Up Addition

- Question: how might we add the following, in decimal?

```
 986
+123
----
   ?
```

	Carry: 1		
		8 +2 -- 0	6 +3 -- 9

Building Up Addition

- Question: how might we add the following, in decimal?

```
  986
+123
----
    ?
```

	1	8	6
	9	+2	+3
	+1	--	--
	--	0	9
	?		

Building Up Addition

- Question: how might we add the following, in decimal?

```
 986
+123
----
   ?
```

Carry: 1	1 9 +1 -- 1	8 +2 -- 0	6 +3 -- 9

Building Up Addition

- Question: how might we add the following, in decimal?

```
  986
 +123
 ----
    ?
```

<pre> 1
 +0
 --
  1</pre>
	<pre> 1
	  9
	 +1
	 --
	  1</pre>
		<pre> 8
		 +2
		 --
		  0</pre>
			<pre> 6
			 +3
			 --
			  9</pre>

Core Concepts

- We have a “primitive” notion of adding single digits, along with an idea of *carrying* digits
- We can build on this notion to add numbers together that are more than one digit long

Now in Binary

- Arguably simpler - fewer one-bit possibilities

<pre> 0
+0
--
  ?</pre>
	<pre> 0
	+1
	--
	  ?</pre>
		<pre> 1
		+0
		--
		  ?</pre>
			<pre> 1
			+1
			--
			  ?</pre>

Now in Binary

- Arguably simpler - fewer one-bit possibilities

0 +0 -- 0	0 +1 -- 1	1 +0 -- 1	1 +1 -- 0 Carry: 1

Chaining the Carry

- Also need to account for any input carry

0 0 +0 -- 0	0 0 +1 -- 1	0 1 +0 -- 1	0 1 +1 -- 0 Carry: 1
1 0 +0 -- 1	1 0 +1 -- 0 Carry: 1	1 1 +0 -- 0 Carry: 1	1 1 +1 -- 1 Carry: 1

Adding Multiple Bits

- How might we add the numbers below?

```
  011
 +001
------
```

Adding Multiple Bits

- How might we add the numbers below?

```
   0
 011
+001
------
```

Adding Multiple Bits

- How might we add the numbers below?

```
  10
 011
+001
------
   0
```

Adding Multiple Bits

- How might we add the numbers below?

```
  110
  011
 +001
------
   00
```

Adding Multiple Bits

- How might we add the numbers below?

```
 0110
  011
 +001
------
  100
```

Adding Multiple Bits

- How might we add the numbers below?

$$\begin{array}{r} \boxed{0}110 \\ 011 \\ +001 \\ \text{------} \\ \boxed{100} \end{array}$$

Output Carry Bit

Result Bits

Another Example

```
  111
 +001
------
```

Another Example

```
    0
  111
 +001
------
```

Another Example

```
  10
 111
+001
------
   0
```

Another Example

```
  110
  111
 +001
------
   00
```

Another Example

```
 1110
  111
 +001
------
  000
```

Output Carry Bit

Result Bits

Output Carry Bit Significance

- For unsigned numbers, it indicates if the result did not fit all the way into the number of bits allotted
- May be an error condition for software

Signed Addition

- Question: what is the result of the following operation?

```
 011
+011
----
   ?
```

Signed Addition

- Question: what is the result of the following operation?

```
 011
+011
----
0111
```

Overflow

- In this situation, *overflow* occurred: this means that both the operands had the same sign, and the result's sign differed

```
 011
+011
----
 110
```

- Possibly a software error

Overflow vs. Carry

- These are **different ideas**
  - Carry is relevant to **unsigned** values
  - Overflow is relevant to **signed** values

`111`
`+001`
`----`
`000`
No Overflow;
Carry
	`011`
	`+011`
	`----`
	`111`
	Overflow;
	No Carry
		`111`
		`+100`
		`----`
		`011`
		Overflow;
		Carry
			`001`
			`+001`
			`----`
			`010`
			No Overflow;
			No Carry

Subtraction

Subtraction

- Have been saying to invert bits and add one to second operand
- Could do it this way in hardware, but there is a trick

```
 001
-001
----
   ?
```

Hint: these two questions are equivalent

```
 001
+111
----
   ?
```

Subtraction Trick

- Assume we can cheaply invert bits, but we want to avoid adding twice (once to add 1 and once to add the other result)
- How can we do this easily?

Subtraction Trick

- Assume we can cheaply invert bits, but we want to avoid adding twice (once to add 1 and once to add the other result)
- How can we do this easily?
    - Set the initial carry to `1` instead of `0`

Subtraction Example

```
 0101
-0011
 ----
```

Subtraction Example

```
 0101
-0011
 ----
```

Invert `0011` $\longrightarrow$

Subtraction Example

```
 0101
-0011
 ----
```

$0011 \xrightarrow{\textbf{Invert } 0011} 1100$

Subtraction Example

```
 0101
-0011
 ----
```

Invert `0011` → `1100` Equivalent to →

Subtraction Example

```
 0101                           1
-0011   Invert 0011              0101
 ----  ------------> 1100 Equivalent to +1100
                         ------------>  ----
```

Subtraction Example

```
 0101
-0011
 ----
```

Invert 0011 $\longrightarrow$ 1100 **Equivalent to** $\longrightarrow$

```
11011
 0101
+1100
 ----
 0010
```

Multiplication (if time)

Multiplication

- For simplicity, we will only consider positive values here
- A number of different algorithms exist; we will only look at one of them

Central Idea

- Accumulate a *partial product*: the result of the multiplication as we go on
  - Computed via a series of additions
- When we are finished, the partial product becomes the final product (the result)
- Build off of addition and multiplication of a single digit (much like with addition)

Decimal Algorithm

- Let `P` be the partial product, `M` be the multiplicand, and `N` be the multiplier
- Initially, `P` is 0
- If `N` is `0`, then `P` = the result
- If not, then `P += ` (the rightmost digit of `N`) times `M`
- Shift `N` right once, and `M` left once
- Repeat

Example

- Performing 803 * 151

Example

- Performing 803 * 151

P	M	N

Example

- Performing 803 * 151

P	M	N
0	803	151

Initially `P = 0`,
`N` = multiplicand
`M` = multiplier

Example

- Performing 803 * 151

P	M	N
0	803	151

N **is not** 0

Example

- Performing 803 * 151

P	M	N
0	803	151
803		

P += (the rightmost digit of N) times M

Example

- Performing 803 * 151

P	M	N
0	803	151
803	8030	15

Shift `N` right once, and `M` left once

Example

- Performing 803 * 151

P	M	N
0	803	151
803	8030	15

N **is not** 0

Example

- Performing 803 * 151

P	M	N
0	803	151
803	8030	15
40953		

`P += ` (the rightmost digit of `N`) times `M`

Example

- Performing 803 * 151

P	M	N
0	803	151
803	8030	15
40953	80300	1

Shift N right once, and M left once

Example

- Performing 803 * 151

P	M	N
0	803	151
803	8030	15
40953	80300	1

N is not 0

Example

- Performing 803 * 151

P	M	N
0	803	151
803	8030	15
40953	80300	1
121253		

`P` += (the rightmost digit of `N`) times `M`

Example

- Performing 803 * 151

P	M	N
0	803	151
803	8030	15
40953	80300	1
121253	803000	0

Shift `N` right once, and `M` left once

Example

- Performing 803 * 151

P	M	N
0	803	151
803	8030	15
40953	80300	1
121253	803000	0

N is 0; done

Intuition

- Only looking at rightmost digit of `N`: getting partial product of that digit with the rest
- Shifting `M` left: for each digit of `N` observed, we look one digit deeper in `M` (and result gets correspondingly larger)
- Similar to traditional pencil-and-paper algorithm (which shifts partial products instead)

Why this Algorithm?

- Looks complex...ish
- On binary, things get simpler. Why?

- Initially, `P` is 0
- If `N` is `0`, then `P` = the result
- If not, then `P  +=` (the rightmost digit of `N`) times `M`
- Shift `N` right once, and `M` left once
- Repeat

Why this Algorithm?

- Looks complex...ish
- On binary, things get simpler. Why?

- Initially, `P` is 0
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- Repeat

Simplified Binary Algorithm

- Initially, `P` is 0
- If `N` is `0`, then `P` = the result
- If not, then `P += ` (the rightmost digit of `N`) times `M`
- Shift `N` right once, and `M` left once
- Repeat

Simplified Binary Algorithm

- Initially, `P` is 0
- If `N` is `0`, then `P` = the result
- If the rightmost digit if `N` is `1`:
    - `P += M`
- Shift `N` right once, and `M` left once
- Repeat

Dealing with Negative Numbers

- Can still be done, but we need extra logic
  - Negative times negative is a positive, positive and a negative is a positive...
- Not fundamentally harder, and showing this extra detail just complicates things in an uninteresting way